440.646 Economics of Derivatives Instructor: John Teall Spring Term 2022

Mid-term Quiz: Practice Version I

1. A well-known test or illustration of the Law of One Price is the "Big Mac Standard" popularized by *The Economist*. MacDonald's Corporation's Big Mac hamburgers are generally regarded to be more or less identical all over the world (excepting India). If the Law of One Price holds, then the Big Mac should sell for the same price in each country after adjusting for exchange rates. However, the prices of Big Macs vary widely after adjusting for exchange rates. Thus, the Law of One Price does not seem to hold with respect to Big Macs. Why doesn't the Law of One Price hold with respect to "Big Macs?"

2.a. Find the value of a one-month European call with exercise price \$60 if the stock price S_T on the expiration date *T* is a random variable on a risk neutral probability space \mathbb{Q} with its uniform density function given by:

$$q(S_T) = \begin{cases} \frac{1}{60} & ,40 \le S_T \le 100\\ 0 & ,elsewhere \end{cases}$$

Assume that the per-annum continuous time risk free rate of interest is 3.5%.

b. What is the current value of a \$1,000 face value coupon riskless bond under the circumstances presented in part a above?

c. Suppose that a stock C is always worth twice as much as the stock referred to in part a above. What would be the value of a call on this stock if its exercise price were \$120?

3. BOS Company stock will pay off either 10, 20 or 30 next year. A call on this stock with an exercise price equal to 25 currently sells for 2; a put with the same exercise price sells for 7. The riskless rate of return is .10. What is the current value of the stock?

4. Suppose that the evolution of a riskless bond's price is modeled by the following equation:

$$\frac{dB_t}{dt} = rB_t$$

The term r > 0 represents the bond's drift or mean instantaneous rate of return. Let the initial bond price $B_0 > X$, where X is the exercise price of a call option on that bond that expires at time *T*. Write a simplified equation to value this call in a risk neutral environment.

5. Consider a bond whose interest rate changes continuously with time, with $B_0 = \$100$, and accruing interest compounded continuously at a rate of r(t) = .05 - .001t. Find the bond's value at time t = 10.

6. Suppose that the evolution of short-term interest rates is described by the differential: $dr = .02 \frac{\sqrt{r}}{\sqrt{t}} dt$ with initial value .0225. Find the solution for the short-term interest rate and, in particular, the short-term interest rate at time 25.

7. Suppose that the price of stock now is $S_0 = \$1$, and its price in 3 months is to be uniformly distributed on the interval [.8, 1.25]. The forward price of a non-dividend-paying stock is often

assumed to derive from the riskless rate of return r: $(F = S_0 e^{rt})$. Given this assumption, find the probability that the stock price S_t will be greater than its forward price $(F = S_0 e^{rt} < S_t)$ in t=3 months if the risk-free rate is 5%.

8. The owner of a cash-or-nothing call, a type of binary call, has the right to receive a specified payment should the underlying asset value (S_T) exceed the exercise price (X) of the call on its expiration date (T). No exercise money is actually paid to call-writer. If the expiration-date stock price is less than the option exercise price, the terminal value of the call is zero. A *digital call* is a type of cash-or-nothing option that pays exactly \$1 if and only if the expiration underlying stock price exceeds the exercise price of the call.

a. Write a formula to value a digital call. Assume that all Black-Scholes assumptions apply in this economy where all investors are risk-neutral. The expected return on the stock equals r_f , the underlying stock variance is σ^2 and the underlying stock pays no dividends.

b. Suppose that a 9-month digital call with an exercise price X = 20 trades on a non-dividend paying stock with a current value $S_0 = 25$. If the riskless return rate is currently $r_f = .20$ and the standard deviation σ of underlying stock returns is .4, what is the current value (we'll call it c_0) of this digital call?

9. Suppose that the log of the stock price relative follows the zero-drift Brownian motion process below where random variable z is distributed normally with parameters (0,1):

$$ln\frac{S_T}{S_t} = \sigma Z \sqrt{T}$$

with t < T. Now, suppose that $S_t = X$. Derive a formula to find the probability *P* that $S_T > X$. What is the numerical value of this probability?

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Mid-term Quiz Solutions: Practice Version I

1. First, Big Macs are not easily exported from countries where they are underpriced or easily imported where they are overpriced; they are perishable and not easily transportable. This means that the proceeds of the sale of an overpriced Big Mac cannot necessarily be used to purchase an underpriced Big Mac. Furthermore, the Law of One Price does not hold when there are differences in taxes, subsidies, labor and other production costs. The Law of One Price requires the absence of market frictions. Using the language of international finance (not a prerequisite for this course), the concept of purchase power parity does not always hold, especially for manufactured goods and for services.

2. Outcomes one through five in a single-period framework correspond to elements in the following probability vectors that exist in \mathbb{P} and in \mathbb{Q} spaces:

 $\mathbb{P} = [0, .1, .21, .29, .4]^{\mathrm{T}}$

 $\mathbb{Q} = [0, .4, .3, .2, .1]^{\mathrm{T}}$

Thus, for example, the probability of outcome 1 is zero under both \mathbb{P} and \mathbb{Q} .

- a. Are \mathbb{P} and \mathbb{Q} equivalent probability measures?
- b. If the current riskless return rate equals 10%, what is the current value of a put option on a stock with the following payoff vector under these same 5-outcome risk-neutral measures with ℚ: [20, 30, 40, 50, 60]^T? You should assume that the put has an exercise price equal to 35.
- c. Suppose that a futures contract trades on the stock in part b. What is the current futures price on this contract?
- d. Suppose that there is a call with an exercise price of 35 trading on the stock from part b. What is the expected risk-neutral value of this call contingent on it being exercised?
- e. Consider the stock for which the payoff vector is given in part b. If one were to use the riskless one-year bond as the numeraire for pricing purposes, what would be the current stock price under its equivalent martingale measure based on the equivalent probability measure Q? (Make sure that you denominate your final numerical answer in terms of either the correct number of dollars or riskless bonds.)

3. This question can be answered in any one of four rather similar ways. Any one of these should suffice. First, put-call parity can be rewritten to obtain the stock price as follows:

$$S_0 = c_0 + X/(1+r) - p_0 = 2 + 25/1.1 - 7 = 17.7275$$

Second, and similarly, the following payoff matrix can be used to obtain the stock price:

$$\begin{bmatrix} 0 & 15 & 1 \\ 0 & 5 & 1 \\ 5 & 0 & 1 \end{bmatrix}$$

c **p** $\mathbf{r}_{\rm f}$

Invert the matrix and use it to pre-multiply the stock payoff vector to obtain the replicating weights:

$$\begin{bmatrix} .1 & -.3 & .2 \\ .1 & -.1 & 0 \\ -.5 & 1.5 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 25 \end{bmatrix}$$

Hence, the current value of the stock is $1 \times 2 - 1 \times 7 + 25 \times .9091 = 17.7275$. Third, pure security prices can be obtained from potential payoffs and current security prices as follows:

$$\begin{bmatrix} \psi_1 & \psi_2 & \psi_3 \end{bmatrix} \begin{bmatrix} 0 & 15 & 1 \\ 0 & 5 & 1 \\ 5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & .9091 \end{bmatrix}$$
$$\begin{bmatrix} .1 & -.3 & .2 \\ .1 & -.1 & 0 \\ -.5 & 1.5 & 0 \end{bmatrix} = \begin{bmatrix} \psi_1 & \psi_2 & \psi_3 \end{bmatrix} = \begin{bmatrix} .44545 & .06365 & .4 \end{bmatrix}$$

Thus, we obtain the current stock price from pure security prices and the stock payoff vector as follows:

$$\begin{bmatrix} .44545 & .06365 & .4 \end{bmatrix} \begin{bmatrix} 10\\ 20\\ 30 \end{bmatrix} = 17.7275$$

Fourth, we can obtain the equivalent martingale from pure security prices then value the stock as follows (Note: $B_0 = .9091$):

$$q_1 = .44545/.9091 = .49; q_2 = .06365/.9091 = .07; q_3 = .4/.9091 = .44$$

 $E_Q[S_0] = [10q_1 + 20q_2 + 30q_3] = 19.5 = S_0/B_0; S_0 = 17.7275$

4. This is a fairly simple example because the bond is riskless. Nevertheless, we separate and solve the differential equation representing the bond's price path as follows:

$$\frac{dB_t}{B_t} = rdt \qquad \int \frac{dB_t}{B_t} = \int rdt \qquad \ln B_T + k_1 = rT + k_2$$
$$e^{\ln B_T} = e^{rT + K} \qquad B_T = e^{rT}e^K \qquad B_T = B_0e^{rT}$$

Since the bond is riskless, its value will certainly exceed *X* at time *T*, and the riskless return is also the discount rate on the bond and exercise money, the call value is:

$$c_T = [B_0 e^{rT} - X]e^{-rT} = B_0 - Xe^{-rT}$$

5. The bond's value at time t=10 is given by $B(10) = 100e^{\left(\int_0^{10} r(t)dt\right)}$: $B(10) = 100e^{\left(\int_0^{10} (.05 - .001t)dt\right)} = 100e^{(.05t - .0005t^2|_0^{10})} = 100e^{.45} = 156.83$

6. Observe that the given differential is separable:

$$\frac{1}{\sqrt{r}}dr = .02\frac{1}{\sqrt{t}}dt.$$

Integrate both sides of the equation:

$$\int r^{-1/2} dr = .02 \int t^{-1/2} dt$$
$$2r^{1/2} = .04t^{1/2} + C$$

Since r(0) = .0225, then: $2\sqrt{.0225} = 0 + C$, and so C = .3. Solving for r(t) results in:

$$r(t) = \left(.02\sqrt{t} + .15\right)^2.$$

The interest rate at time t = 25 is: r(25) = .0625.

7. First notice the forward price will be $1 \times e^{.05 \times .25} = e^{.0125}$. The density function

$$f(x) = \frac{1}{1.25 - .8} = \frac{1}{.45}$$

implies that:

$$F(X > e^{.0125}) = \int_{e^{.0125}}^{1.25} \frac{1}{.45} dx = \frac{1}{.45} (1.25 - e^{.0125}) = 0.5276, \text{ or } 52.76\%$$

8.a. The digital call provides for the option holder to receive \$1, with probability $N(d_2)$. Thus, the future expected value of this digital call is simply $1 \times N(d_2)$. Multiplying this conditional expected future value by its probability $N(d_2)$, and discounting by multiplying by e^{-rfT} yields the following formula for the asset-or-nothing call value:

$$c_0 = e^{-r_f T} N(d_2)$$

To develop a better intuition of this model, carefully read the section in Chapter 7 on Option Pricing: A Heuristic Derivation of the Black Scholes Model. This illustration is quite a bit easier than most illustrations related to the section in Chapter 7.

b. Plug into the formula derived in part a to obtain $d_1 = 1.25$, $d_2 = .904$, $N(d_2) = .817$ and $c_0 = .703$.

9. Since a zero-drift Brownian motion process is a martingale, $E[S_T|S_t] = S_t$ with (t < T), with a 50% probability that S_T will exceed S_t . Alternatively, we can solve as follows:

 $P(S_T > S_t) = P(S_t e^{\sigma \sqrt{T}Z} > S_t) = P(e^{\sigma \sqrt{T-t}Z} > 1) = P(Z > 0) = N(0) = .5$

The key intuition here draws from the premises that $S_t = X$ and that there is zero drift. Without drift, the expected value of a Brownian motion process is its prior value since Brownian motion is a martingale.

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Mid-term Quiz: Practice Version II

1.a. How does novation affect counterparty risk in options markets? If counterparty risk in options markets is unaffected by novation, explain why it isn't.

b. Suppose that two potential counterparties in trade wish to take opposite positions (long and short) in the same asset. There are no futures markets on this asset (and no clearinghouses), so they would need to transact in forward markets. The two potential counterparties are not willing to assume credit or counterparty risk on the other. How might a mutually beneficial transaction be able to execute between these two parties in forward markets so that neither counterparty incurs the credit risk of the other?

2. A put and a call are selling for \$5 each on a share of stock currently worth \$50. Both the put and call expire in one year and have exercise prices equal to \$50. The market for stocks and options are perfectly efficient, with no-arbitrage pricing evident.

a. What is the riskless return rate in this economy?

b. How would your answer to part a of this question change if investors were strongly risk averse?

c. In this same economy, suppose that the spot price of gold is \$1,800 per ounce. What is the futures price of an ounce of gold assuming that the Expectations Hypothesis for futures pricing holds? Ignore part d of this question to answer this part c.

d. Now, assume a single exception to perfect market efficiency, with the annual cost of storing gold being \$2 per ounce. However, the riskless interest or return rate is still consistent with the correct answer for part a. Would this futures market for gold more likely be in contango, backwardation, both or neither?

3. A stock that is currently selling for S_0 can either increase in price to uS_0 at time 1, remain the same at S_0 or decline in price to dS_0 . At time 2, the price could either increase by multiplicative rate *u* from its time 1 value, remain the same as its time 1 value or decrease to proportion *d* of its time 1 value. Write all of the elements of the time 1 filtration for this process.

4. Let $\{r_t, t \ge 0\}$ (the return on a stock) be an arithmetic Brownian motion.

a. Suppose that r_t is made up of two components, an instantaneous drift with expected value μ = .05 and a variance σ^2 = .25. What is the probability that r_5 is between .3 and .5?

b. Suppose that the price of a stock follows a geometric Brownian motion process. Suppose that the stock's initial value $S_0 = 1$, its instantaneous drift r_t has an expected value $\mu = .05$ per year and an annual variance $\sigma^2 = .25$. What is the probability that the stock is worth more than 2 in five years $P[S_5 > 2]$?

c. Is the return process for this stock a martingale?

5. In a discreet, two-period, perfectly efficient market, a stock is selling for \$1 per share. In each of the two one-year periods in this economy, the stock's price will either double or drop in price

by 40%. All riskless bonds will yield 10% each year; that is, the yield curve is flat. You may assume that markets are efficient and allow for no-arbitrage pricing.

a. What is the time-zero (now) no-arbitrage market value of a 2-year call in this economy if its exercise price equals 2?

b. What would be the risk-neutral probability associated with an increase in the stock price during the first period followed by a decrease in the second?

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Mid-term Quiz Solutions: Practice Version II

1.a. Novation, the process by which a clearinghouse assumes counterparty settlement obligations, functions in much the same way in options markets as in other markets. Essentially, novation transfers credit or counterparty risk from participants in option market transactions to the clearinghouse, frequently owned and backed by an exchange. Each clearing house member, including brokers and certain traders serve as back-up underwriters of counterparty risk. Thus, clearinghouses mitigate counterparty risk by assuming settlement obligations themselves such they become the only source of counterparty risk.

b. Such transactions occur regularly in forward markets. Usually, a major bank or other reputable financial institution steps in to serve as counterparty (including settlement obligations) to each of the two transacting parties such that the financial institution assumes the risk on both sides of the forward market transaction. Similar scenarios also regularly occur in swap and OTC options market transactions. Thus, clearinghouses are not essential to novation; they are simply the natural counterparty for exchange-based transactions.

2.a. **0**, by put-call parity. Put-call parity holds in this case only when r = 0.

b. It would not change. Risk aversion is irrelevant in a risk-neutral economy because of noarbitrage pricing.

c. \$1,800 since the riskless interest rate equals zero.

d. **backwardation**: In a zero-interest rate market, futures prices are likely to exceed spot prices in the presence of storage costs but no other market inefficiencies.

3. The time 1 filtration is written as follows:

b.

 $\mathfrak{I}_{1} = \{ \emptyset, \{ (uS_{0}, uuS_{0}), (uS_{0}, uS_{0}), (uS_{0}, udS_{0}) \}, \{ (S_{0}, uS_{0}), (S_{0}, S_{0}), (S_{0}, dS_{0}) \}, \{ (dS_{0}, duS_{0}), (dS_{0}, dS_{0}), (dS_{0}, ddS_{0}) \}, \Omega \}.$

4.a. First, note that observations from an arithmetic Brownian motion process are normally distributed, in this case with $\mu = .05$ and a standard deviation $\sigma = .5$. We calculate the probability of the range as follows:

 $\begin{aligned} &\Pr\{.3 \le r_5 \le .5\} = N([\text{upper limit} - \mu T]/[T^{.5}\sigma]\} - N([\text{lower limit} - \mu T]/[T^{.5}\sigma]\} \\ &= N([.5-5 \times .05]/[5^{.5 \times .5}]) - N([.3-(5 \times .05)]/[5^{.5 \times .5}]) = .588468 - .517835 =$ **.070633** $\\ &\Pr\{ln(S_5/S_0) > 2\} = 1 - N([ln(2) - T \times (\mu - .5\sigma^2)]/[T^{.5}\sigma]\} \\ &= 1 - N([.63147 - 5 \times (.05 - .5 \times .25)]/[5^{.5 \times .5}]) = 1 - N(.95538) =$ **.169693** $\end{aligned}$

c. Yes: The return process is a martingale because it reflects a constant drift so that it's expected value $E[r_t] = \mu = .05$ for all *t*. However, since the drift is positive, the stock price process is a submartingale. Also, since this return process follows arithmetic Brownian motion, and arithmetic Brownian motion is always a martingale, the return process must be a martingale.

5.a. It's easy to calculate that bond prices are $1/(1+.1)^2 = .826446$ and share prices are simply 1. One year, bond prices will be 1/(1+.1) = .9091, and share prices will be either 2 or .6. Thus, first-period pure security prices are calculated as follows:

$$\begin{bmatrix} 2 & .6 \\ .9091 & .9091 \end{bmatrix} \begin{bmatrix} \psi_{0,1;u} \\ \psi_{0,1;d} \end{bmatrix} = \begin{bmatrix} 1 \\ .826446 \end{bmatrix},$$

which we can solve by inverting the 2×2 matrix above and rewriting as follows:

$$\begin{bmatrix} .714286 & -0.47143 \\ -.71429 & 1.571429 \end{bmatrix} \begin{bmatrix} 1 \\ .826446 \end{bmatrix} = \begin{bmatrix} \psi_{0,1;u} \\ \psi_{0,1;d} \end{bmatrix} = \begin{bmatrix} 0.324675 \\ 0.584416 \end{bmatrix}$$

Second-period pure security prices are calculated as follows:

$$\begin{bmatrix} \psi_{0,2;u,u} \\ (\psi_{0,2;u,d} + \psi_{0,2;d,u}) \\ \psi_{0,2;d,d} \end{bmatrix} = \begin{bmatrix} \psi_{1,2;u,u} & 0 \\ \psi_{1,2;u,d} & \psi_{1,2;d,u} \\ 0 & \psi_{1,2;d,d} \end{bmatrix} \begin{bmatrix} \psi_{0,1;u} \\ \psi_{0,1;d} \end{bmatrix}$$
$$\begin{bmatrix} \psi_{0,2;u,u} \\ (\psi_{0,2;u,d} + \psi_{0,2;d,u}) \\ \psi_{0,2;d,d} \end{bmatrix} = \begin{bmatrix} .3247 & 0 \\ .5844 & .3247 \\ 0 & .5844 \end{bmatrix} \begin{bmatrix} .3247 \\ .5844 \end{bmatrix} = \begin{bmatrix} .1054 \\ .3795 \\ .3415 \end{bmatrix}$$

Since the call only exercises in the event of the 1st outcome ($S_2 = 4$), the value of the call is simply ($S_2 - X$)× $\psi_{0,2;u,u} = (4-2)\times.1054 = .2108$. There are a number of other ways to solve this problem as well.

b. The pure security price associated with this event is $\psi_{0,2;u,d} = .5844 \times .3247 = .18975$. To obtain the risk-neutral probability for outcome 2 for the second period, simply multiply this pure security price by 1.1^2 to obtain $.18975 \times 1.1^2 = .22959 = q_2$.

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Final Exam: Practice Version 1

1. A binomial option pricing model prices an option based on two potential outcomes and two priced securities (e.g., a riskless asset and the stock underlying the option). Now, suppose that a particular market consists of two priced securities: a riskless asset paying 10% and shares of common stock currently selling for 10. In addition, there are calls written on these shares with an exercise price of X=8. This market exists in a single time period framework with *three* rather than two potential outcomes. The stock, which is currently selling for 10 will either decrease to 5 next year, remain unchanged at 10 or increase to 15. Physical (or real) outcome probabilities are unknown. In this market, can a "trinomial" option pricing model based on construction of hedge portfolios be derived to value the option with an exercise price X=8? If so, how? If not, why not?

2. A stock currently selling for \$40 in a one time period binomial environment has a one-year call option trading on it with an exercise price equal to \$40. The current Treasury bill rate equals .10 and the probability of an upward price movement is projected to be .52. What are the projected multiplicative upward and downward price movements? Your answer should be within .05.

3. The following Itô process describes the price of a given stock:

$$dS_t = .1(S_t, t) + .5(S_t, t)dz$$

a. What is the solution to this stochastic differential equation?

b. Suppose that the stochastic differential equation above applies for a one-year holding period. What are the expected value and variance of the log of price relative for this stock over a one-month period?

4. Estimate the implied Black-Scholes volatility (standard deviation) for the following call: $c_0 = 10$, $S_0 = 25$, T = .5, X = 35 and $r_f = .10$. Your answer should be within .01.

5. An investor purchases a one-year American call on a stock that currently sells for \$75 a share with $\sigma = .15$. The stock is expected to go ex-dividend in 3 months with a dividend of \$4. The exercise price of the call is \$85. The riskless return rate is 4%.

a. Based on Black's Pseudo-American call model, what would be the price of the call?

b. Based on the Roll-Geske-Whaley model, what would be the price of the call?

6. What does an option *vega* measure? Suppose that an option speculator has an option portfolio on a single stock. The speculator believes that the stock risk is going to increase substantially – more so than is reflected in option market prices. Should he increase or decrease his portfolio vega? Should the speculator buy or sell calls to change his portfolio vega? Should he buy or sell puts to change his portfolio vega?

7. If $S_t = (20 + .1t)e^{.05Z_t}$, find dS_t .

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Final Exam Solutions: Practice Version 1

1. A "trinomial" option pricing model cannot be derived because a hedge portfolio cannot be constructed in a one time period, three-potential outcome scenario. That is, no single value for α satisfies:

$$\alpha u P_0 - c_u = \alpha P_0 - c_{no \ change} = \alpha dP_0 - c_d$$

$$\alpha 15 - 7 = \alpha 10 - 2 = \alpha 5 - 0$$

Similarly, markets are not complete. There do not exist three priced securities (only two) whose payoff vectors span the 3-dimensional payoff space. The option could be priced if another third priced security existed in this market, as long as its price was not inconsistent with the prices of the other two priced securities.

2. Solve the following for *u*:

$$p = \frac{e^{r_f T} - d}{u - d} = .52 = \frac{e^{.1 \times 1} - 1/u}{u - 1/u}$$

Thus, the multiplicative upward and downward movements for the stock are projected to be 1.5 and .6667, respectively. Substitute 1.5 for u to verify that p = .52 or solve algebraically for u.

3.a.
$$S_T = S_0 e^{\left[\left(.1 - \frac{5^2}{2}\right)T + .5z_r\right]}$$

b. $E\left[\ln\frac{S_{1/12}}{S_0}\right] = \mu T - \frac{1}{2}\sigma^2 T = \left(.1 - \frac{.5^2}{2}\right) \cdot 1/12 = -.0020833$

4. Try an initial guess for standard deviation equal to, for example, .5. It doesn't matter much what your initial trial estimate is. This .5 estimate results in a call value equal to 1.93. This call value is much too small, so increase the standard deviation estimate. Try a much larger estimate, say $\sigma = 2$. This estimate results in a call value that is a little large, but reasonably close. So, we try a smaller estimate, iterating until our trial value seems close enough. Ultimately, we arrive at an estimate within .01 of the correct standard deviation of 1.798.

5.a. First, we check to see if the dividend is greater than the time value associated with the exercise money:

$$4 > 85(1 - e^{-.04(1 - .25)}) = 2.512.$$

Since the dividend is greater, the call should be exercised early. Next, we calculate the call value assuming it trades just before the ex-dividend date:

$$d_{1} = \frac{\ln(75/85) + \left(.04 + \frac{1}{2} \cdot 15^{2}\right)(.25)}{.15\sqrt{.25}} = -1.498009, d_{2} = -1.498009 - .15\sqrt{.25} = -1.573009$$

$$c_{0}^{*} = 75N(-1.498009) - 85e^{-.04 \times .25}N(-1.573009) = .16088$$
Now, we calculate the call assuming the antion is held until it exprises:

Now, we calculate the call assuming the option is held until it expires:

$$d_{1} = \frac{ln\left(\frac{75-4e^{-.04\times.25}}{85}\right) + \left(.04+\frac{1}{2}\cdot15^{2}\right)(1)}{.15\sqrt{1}} = -.8544064, d_{2} = -.8544064 - .15\sqrt{1} = -1.0044064$$

$$c_{0}^{**} = (75-4e^{-.04\times.25})N(-.8544064) - 85e^{-.04\times1}N(-1.0044064) = 1.08502$$
The analysis of the call is the maximum of the two call values:

The value of the call is the maximum of the two call values:

 $c_0 = Max(.16088, 1.08502) =$ \$1.08502.

b. First, we need to find the critical stock price S_{tD}^* so that

$$S_{tD}^* N\left(\frac{\ln(S_{tD}^*/85) + \left(.04 + \frac{1}{2}.15^2\right)(1 - .25)}{.15\sqrt{1 - .25}}\right) - 85e^{-.04 \times (1 - .25)} N\left(\frac{\ln\left(\frac{S_{tD}^*}{85}\right) + \left(.04 - \frac{1}{2}.15^2\right)(1 - .25)}{.15\sqrt{1 - .25}}\right) = S_{tD}^* + 4 - 85.$$

Using an approximation method such as the method of bisection, one finds that $S_{tD}^* = 90.86189$. As long as the stock price on the ex-dividend date $t_D = .25$ exceeds \$90.86189, then the call will be exercised early. To determine the value of the call:

$$d_{1} = \frac{ln\left(\frac{75-4e^{-.04\times.25}}{85}\right) + \left(.04 + \frac{1}{2}.15^{2}\right)(1)}{\frac{.15\sqrt{1}}{90.86189}} = -.8544064 , d_{2} = -.8544064 - .15\sqrt{1} = -1.0044064 , d_{2} = -.8544064 - .15\sqrt{1} = -.8544064 - .15\sqrt{1} = -.8544064 + .15\sqrt{1} = -.8540$$

The value of the call is:

$$\begin{split} c_0 &= (75 - 4e^{-.04 \times .25})N(-3.110505) + (75 - 4e^{-.04 \times .25})M(-.8544064, 3.110505, -\sqrt{\frac{.25}{1}}) \\ &- 85e^{-.04 \times 1}M\left(-1.0044064, 3.185505, -\sqrt{\frac{.25}{1}}\right) - (85 - 4)e^{-.04 \times .25}N(-3.185505) \\ &= 1.08526. \\ d_1 &= \frac{\ln{(\frac{40}{37})} + (.06 - .04 + \frac{1}{2}.12^2)(.5)}{.12\sqrt{.5}} = 1.079063, d_2 = 1.079063 - .12\sqrt{.5} = .9942103 \\ c_0 &= 40e^{-.04 \times .5}N(1.079063) - 37e^{-.06 \times .5}N(.9942103) = 3.54858. \\ p_0 &= 3.54858 + 37e^{-.06 \times .5} - 40e^{-.04 \times .5} = .247118. \\ d_1 &= \frac{\ln{(\frac{.14}{12})} + (.04 - .05 + \frac{1}{2}.2^2)(.25)}{.2\sqrt{.25}} = 1.566507, d_2 = 1.566507 - .2\sqrt{.25} = 1.466507 \\ c_0 &= .14e^{-.05 \times .25}N(1.566507) - .12e^{-.04 \times .25}N(1.466507) = .019816. \end{split}$$

6. Vega measures the sensitivity of the option price to the underlying stock's standard deviation of returns. If an investor believed that stock risk was going to increase, he should increase the vega of his option portfolio. This means that he should buy both calls and puts to increase his portfolio vega.

7. By the general power rule:

By Itô's Lemma:

$$dS_t = .1e^{.05Z_t}dt + (20 + .1t)d(e^{.05Z_t}) + .1d(e^{.05Z_t})dt.$$

$$d(e^{,05Z_t}) = .05e^{,05Z_t}dZ_t + .00125e^{,05Z_t}dt.$$

Since $dZ_t dt$ and $(dt)^2$ are negligible, then: $dS_t = .1e^{.05Z_t} dt + (20 + .1t)(.05e^{.05Z_t} dZ_t + .00125e^{.05Z_t} dt)$ $= (.125 + .000125t)e^{.05Z_t} dt + (1 + .005t)e^{Z_t} dZ_t.$

440.646 Economics of Derivatives Instructor: John Teall Spring Term 2022

Final Exam: Practice Version 2

1. Write the solution to the following stochastic differential equation:

 $dS_t = (\mu + at)S_t dt + \sigma S_t dZ_t$

with initial value $S(0) = S_0$.

2. Smedley Company stock is currently selling for \$40 per share. Its historical variance of returns is .25, compared to the historical market variance of .10. The current one-year treasury bill rate is 5%. Assume that all of the standard Black-Scholes Option Pricing Model assumptions hold.

- a. What is the current value of a put on this stock if it has an exercise price of \$35 and expires in one year?
- b. What is the probability implied by problem inputs that the value of the stock will be less than \$30 in one year?

3. A share of ACME Corporation currently sells for \$45 with a projected annual return standard deviation of .3. A compound call on a call has an exercise price of \$4 expiring in 3 months. The underlying call has an exercise price of \$40 expiring in 1 year. The riskless return rate is 4%. What is the value of the compound call?

4. Currency options with the following terms are being offered on U.S. dollars (*USD*) denominated in Latvian lat (*LVL*):

Т	=	.5 years		r(f)	=	.1
r(d)	=	.1		S_O	=	LVL1.8/USD
σ	=	.4		X	=	LVL1.8/USD
		-	11			

- a. Evaluate the European call.
- b. Evaluate the European put.
- c. What is the probability that the value of the lat will exceed USD0.5556 in six months?

5. Consider an American call option on a non-dividend-paying stock where the stock price is \$10, strike price is \$7, the stock volatility (σ) is 10%, and the time to maturity (T) is 6 months. Suppose risk-free interest rate is 4% per annum.

- a. What is the price of this call using a two-period tree? Assume u=1.5 and d=.5.
- b. What is the price of this call using the Black-Scholes Option Pricing Model?
- c. Now let $u = e^{\sigma\sqrt{t}}$, $d = \frac{1}{u}$, $q = \frac{e^{rt}-d}{u-d}$, and $t = \frac{T}{n}$, What is the price of the American call option using a two-step tree? (This is called the Cox, Ross & Rubinstein (CRR) method)
- d. What would you conclude about CRR relative to Black-Scholes for an example like this?
- 6. If we define d_1 and d_2 as in part a, then:

$$d_{2} = \frac{ln\left(\frac{S_{0}}{X}\right) + \left(r - \frac{1}{2}\sigma^{2}\right)T}{\sigma\sqrt{T}}$$

We can rewrite this as:

$$d_{2} = \frac{\ln\left(\frac{S_{0}}{X}\right) + \left(r + \frac{1}{2}\sigma^{2}\right)T - \sigma^{2}T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{S_{0}}{X}\right) + \left(r - \frac{1}{2}\sigma^{2}\right)T}{\sigma\sqrt{T}} - \sigma\sqrt{T}$$
$$= d_{1} - \sigma\sqrt{T}.$$

This is the same definition d_2 in part b. This matters for two reasons. First, different books and manuals define d2 as in part a, and others as in part b. Second, it is useful to calculate d_2 independently of d_1 for the purpose of obtaining implied probabilities that the underlying security price will be within certain ranges.

Instructor: John Teall Spring Term 2022

Economics of Derivatives

Final Exam Solutions: Practice Version 2

1. First, write the stochastic differential equation in the form:

$$\frac{dS_t}{S_t} = (\mu + at)dt + \sigma dZ_t.$$

By Itô's Lemma, note that:

$$d(\ln S_t) = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} (dS_t)^2 = \frac{1}{S_t} dS_t - \frac{\sigma^2}{2} dt.$$

Thus:

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$$d(lnS_t) = \left(\mu - \frac{\sigma^2}{2} + at\right)dt + \sigma dZ_t.$$

Integrating both sides of this equation from 0 to T:

$$ln\left(\frac{S_T}{S_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right)T + \frac{1}{2}aT^2 + \sigma Z_T$$

Solving for *S*_{*T*}:

$$S_T = S_0 e^{\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \frac{1}{2}aT^2 + \sigma Z_T\right]}.$$

2. a. d₁= .6172; d₂= .1178; N(d₁) = .7314; N(d₂) = .5469 c₀= 11.058; with put-call parity: p0 = 4.35 b. Use X=30; d₁= .925; d₂= .4245; N(d₂) = .6644 1-N(d₂) = .3356

3. The critical underlying stock price S_{T1}^* must satisfy the equation:

$$c_{u,T1} = S_{T1}^* N\left(\frac{\ln(S_{T1}^*/40) + \left(.04 + \frac{1}{2}.3^2\right)(1 - .25)}{.3\sqrt{1 - .25}}\right)$$
$$- 40e^{-.04(1 - .25)} N\left(\frac{\ln\left(\frac{S_{T1}^*}{40}\right) + \left(.04 - \frac{1}{2}.3^2\right)(1 - .25)}{.3\sqrt{1 - .25}}\right) = 4.$$

Using a desired approximation method such as the method of bisection, one finds that $S_{T1}^* = 38.79585$

Next, we calculate:

$$d_{1} = \frac{ln(45/38.79585) + \left(.04 + \frac{1}{2}.3^{2}\right)(.25)}{.3\sqrt{.25}} = 1.1306614, d_{2} = 1.130661 - .3\sqrt{.25}$$
$$= .9806614$$
$$y_{1} = \frac{ln(45/40) + \left(.04 + \frac{1}{2}.3^{2}\right)(1)}{.3\sqrt{1}} = .67594345, y_{2} = .67594345 - .3\sqrt{1} = .37594345$$

We can now directly calculate the value of the compound call:

$$c_{0,call} = 45M \left(1.1306614, .67594345, \sqrt{\frac{.25}{1}} \right) - 40e^{-.04 \times 1}M \left(.9806614, .37594345, \sqrt{\frac{.25}{1}} \right) - 4 \times e^{-.04 \times .25}N(.9806614) = 5.17512.$$

- 4.a. Notice that the options are denominated in Latvian lat (LVL), not USD. $d_1 = 0.1414; d_2 = -0.1414; N(d_1) = 0.5562; N(d_2) = 0.4438; c_0 = 0.1926$
 - b. $p_0 = Xe^{-rt} + c_0 s_0 = 0.1926$
 - c. Notice now that the options will be denominated in USD rather than LVL. $Pr[LVL_5 > USD0.5556] = Pr[USD_5 < LVL1.8] = 1-N(d_2) = 0.5562$

5.a. First, we need to establish a riskless rate for each period on the two-period lattice. The annual riskless rate was given to be .04. We are working with a 6-month expiration here, which will be divided into two-three-month periods. So, the rate per period is .01, or a little less if you wish to compound it. We solve for the value of the call using the binomial model as follows:

$$q = \frac{1+r-d}{u-d} = \frac{1+.01-.5}{1.5-.5} = \frac{.51}{1} = .51$$

$$uuS_0 = 1.5 \times 1.5 \times 10 = 22.5$$

$$duS_0 = udS_0 = 1.5 \times .5 \times 10 = 2.5$$

$$c_0 = \frac{\sum_{j=0}^{T} \frac{T!}{j!(T-j)!} q^{j}(1-q)^{T-j} MAX[u^j d^{T-j}S_0 - X, 0]}{(1+r)^T}$$

$$c_0 = \frac{2!}{2!0!} \frac{(.5025)^2 (.4975)^0 Max(22.5-7.0) + \frac{2!}{1!1!} (.5025)^1 (.4975)^{1/4} Max(7.5-7.0) + \frac{2!}{0!2!} (.5025)^0 (.4975)^{2} Max(2.5-7.0)}{(1+.01)^2}$$

$$= \frac{.2525(15.5) + 2(.5025)(.4975)(2.5) + .2476(0)}{1.01^2} = \frac{4.1638}{1.0201} = 4.08$$
b. $d_2 = \frac{\ln(\frac{S_0}{X}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(\frac{10}{7}) + (.04 - \frac{1}{2}(.1^2)).5}{.1(\sqrt{5})} = 5.292$

$$d_1 = d_2 + \sigma\sqrt{T} = 5.292 + .1(\sqrt{.5}) = 5.363$$
From the z-table, N(d_1) = N(5.363) = 1 ; N(d_2) = N(5.292) = 1
$$c_0 = S_0 N(d_1) - Xe^{-rt} N(d_2) = 10 \times 1 - 7 \times e^{-.04 \times .5} \times 1 = 3.139$$
c. $u = e^{\sigma\sqrt{t}} = e^{.1(\sqrt{.25})} = 1.0513$

$$d = \frac{1}{t} = \frac{1}{t + t} = 0.9512$$

$$u = 1.0513 \\ uS_0 = 1.0513 \times 10 = 10.513 \\ uS_0 = 1.0513 \times 1.0513 \times 10 = 11.052 \\ dS_0 = .9512 \times 10 = 9.512 \\ udS_0 = duS_0 = 1.0513 \times .9512 \times 10 = 9.048 \\ udS_0 = duS_0 = 1.0513 \times .9512 \times 10 = 10 \\ c_0 = \frac{\sum_{j=0}^{T} \frac{T!}{j!(T-j)!} q^{j}(1-q)^{T-j} MAX[u^{j}d^{T-j}S_0 - X, 0]}{(1+r)^T} \\ c_0 = \frac{\frac{2!}{210!} (.5125)^2 (.4875)^0 Max(11.052 - 7, 0) + \frac{2!}{11!} (.5125)^1 (.4875)^1 Max(10 - 7, 0) + \frac{2!}{012!} (.5125)^0 (.4875)^2 Max(9.048 - 7, 0)}{(1+.01)^2} \\ = \frac{.2627(4.052) + 2(.5125)(.4875)(3) + (.4875^2)(2.048)}{1.01^2} = \frac{3.05}{1.0201} = 2.99$$

d. Calculating the price using the CRR binomial tree model reasonably closely approximates the price using the Black-Scholes model.

6. If we define d_1 and d_2 as in part a, then:

$$d_{1} = d_{2} + \sigma\sqrt{T} = \frac{\ln\left(\frac{S_{0}}{X}\right) + \left(r - \frac{1}{2}\sigma^{2}\right)T}{\sigma\sqrt{T}} + \sigma\sqrt{T}$$
$$= \frac{\ln\left(\frac{S_{0}}{X}\right) + \left(r - \frac{1}{2}\sigma^{2}\right)T + \sigma^{2}T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{S_{0}}{X}\right) + \left(r + \frac{1}{2}\sigma^{2}\right)T}{\sigma\sqrt{T}}.$$

~

Also, since $d_1 = d_2 + \sigma \sqrt{T}$, then $d_2 = d_1 - \sigma \sqrt{T}$. These are precisely the definitions of d_1 and d₂ in part b.