## Johns Hopkins University

440.646 Instructor: John Teall
Economics of Derivatives Fall Term 2022

Sample Quizzes and Exams

What follows are two sample mid-term quizzes and two sample final exams for Economics of Derivatives. Solutions will follow each quiz or exam. I will also place the second version of the sample quiz on the Respondus exam site so that you can experience taking the quiz under actual quiz conditions. The solutions will still be on this pdf document starting at approximately page 9. All of these intended to assist you in the preparation for the mid-term quiz and the final exam for the course.

I suggest that you not look at these practice exams until you are nearing the completion of your preparations for the actual quiz or exam, but this is up to you. If you are familiar with the questions (and solutions) before taking these practice exams under "exam conditions," they will seem quite easy relative to the actual exam. Questions on these practice quizzes and exams were drawn from actual quizzes and exams in prior semesters and each should require approximately 2 hours to complete if you are well-prepared. Solutions, usually worked through in steps, to quiz and exam questions follow the sample exams. These sample exams might be revised as the term progresses to reflect actual course coverage. Again, allow approximately 2 hours to complete each sample exam.

Two versions of the mid-term quiz and its solutions are at the beginning of this pdf document. These two sample quizzes are or will be followed by two versions of the sample final exam.

# Johns Hopkins University

440.646 Instructor: John Teall
Economics of Derivatives Spring Term 2022

Mid-term Quiz: Practice Version 1; Two Hours, Open Book

1. A well-known test or illustration of the Law of One Price is the "Big Mac Standard" popularized by *The Economist*. MacDonald's Corporation's Big Mac hamburgers are generally regarded to be more or less identical all over the world (excepting India). If the Law of One Price holds, then the Big Mac should sell for the same price in each country after adjusting for exchange rates. However, the prices of Big Macs vary widely after adjusting for exchange rates. Thus, the Law of One Price does not seem to hold with respect to Big Macs. Why doesn't the Law of One Price hold with respect to "Big Macs?"

2. BOS Company stock will pay off either 10, 20 or 30 next year. A call on this stock with an exercise price equal to 25 currently sells for 2; a put with the same exercise price sells for 7. The riskless rate of return is .10. What is the current value of the stock?

3. Suppose that the evolution of a riskless bond's price is modeled by the following equation:

$$\frac{dB_t}{dt} = rB_t$$

The term $r > 0$ represents the bond's drift or mean instantaneous rate of return. Let the initial bond price $B_0 > X$, where $X$ is the exercise price of a call option on that bond that expires at time $T$. Write a simplified equation to value this call in a risk neutral environment.

4. Consider a bond whose interest rate changes continuously with time, with $B_0 = \$100$, and accruing interest compounded continuously at a rate of $r(t) = .05 - .001t$. Find the bond's value at time $t = 10$.

5. Suppose that the evolution of short-term interest rates is described by the differential: $dr = .02\frac{\sqrt{r}}{\sqrt{t}}dt$ with initial value .0225. Find the solution for the short-term interest rate and, in particular, the short-term interest rate at time 25.

6. The owner of a cash-or-nothing call, a type of binary call, has the right to receive a specified payment should the underlying asset value ($S_T$) exceed the exercise price ($X$) of the call on its expiration date ($T$). No exercise money is actually paid to call-writer. If the expiration-date stock price is less than the option exercise price, the terminal value of the call is zero. A *digital call* is a type of cash-or-nothing option that pays exactly \$1 if and only if the expiration underlying stock price exceeds the exercise price of the call.

a. Write a formula to value a digital call. Assume that all Black-Scholes assumptions apply in this economy where all investors are risk-neutral. The expected return on the stock equals $r_f$, the underlying stock variance is $\sigma^2$ and the underlying stock pays no dividends.

b. Suppose that a 9-month digital call with an exercise price $X = 20$ trades on a non-dividend paying stock with a current value $S_0 = 25$. If the riskless return rate is currently $r_f = .20$ and the standard deviation $\sigma$ of underlying stock returns is .4, what is the current value (we'll call it $c_0$) of this digital call?

7.a. How does novation affect counterparty risk in options markets? If counterparty risk in options markets is unaffected by novation, explain why it isn't.

b. Suppose that two potential counterparties in trade wish to take opposite positions (long and short) in the same asset. There are no futures markets on this asset (and no clearinghouses), so they would need to transact in forward markets. The two potential counterparties are not willing to assume credit or counterparty risk on the other. How might a mutually beneficial transaction be able to execute between these two parties in forward markets so that neither counterparty incurs the credit risk of the other?

8. In a perfectly efficient capital market, is it possible for a high-risk security to have a higher NPV (Net Present Value: Present value less initial cash investment) than a low risk security? Why or why not?

## Johns Hopkins University

440.646 Instructor: John Teall
Economics of Derivatives Spring Term 2021

Mid-term Quiz Solutions: Practice Version 1

1. First, Big Macs are not easily exported from countries where they are underpriced or easily imported where they are overpriced; they are perishable and not easily transportable. The Law of One Price does not hold when there are differences in taxes, subsidies, labor and other production costs.

2. This question can be answered in any one of four rather similar ways. Any one of these should suffice. First, put-call parity can be rewritten to obtain the stock price as follows:

$$S_0 = c_0 + X/(1+r) - p_0 = 2 + 25/1.1 - 7 = 17.7275$$

Second, and similarly, the following payoff matrix can be used to obtain the stock price:

$$\begin{array}{c}\begin{bmatrix}0 & 15 & 1\\ 0 & 5 & 1\\ 5 & 0 & 1\end{bmatrix}\\ \mathbf{c}\quad\ \mathbf{p}\quad\ \mathbf{r_f}\end{array}$$

Invert the matrix and use it to pre-multiply the stock payoff vector to obtain the replicating weights:

$$\begin{bmatrix}.1 & -.3 & .2\\ .1 & -.1 & 0\\ -.5 & 1.5 & 0\end{bmatrix}\begin{bmatrix}10\\ 20\\ 30\end{bmatrix} = \begin{bmatrix}1\\ -1\\ 25\end{bmatrix}$$

Hence, the current value of the stock is 1×2 - 1×7 + 25×.9091 = 17.7275. Third, pure security prices can be obtained from potential payoffs and current security prices as follows:

$$[\psi_1 \quad \psi_2 \quad \psi_3]\begin{bmatrix}0 & 15 & 1\\ 0 & 5 & 1\\ 5 & 0 & 1\end{bmatrix} = [2 \quad 7 \quad .9091]$$

$$[2 \quad 7 \quad .9091]\begin{bmatrix}.1 & -.3 & .2\\ .1 & -.1 & 0\\ -.5 & 1.5 & 0\end{bmatrix} = [\psi_1 \quad \psi_2 \quad \psi_3] = [.44545 \quad .06365 \quad .4]$$

Thus, we obtain the current stock price from pure security prices and the stock payoff vector as follows:

$$[.44545 \quad .06365 \quad .4]\begin{bmatrix}10\\ 20\\ 30\end{bmatrix} = 17.7275$$

Fourth, we can obtain the equivalent martingale from pure security prices then value the stock as follows (Note: $B_0 = .9091$):

$$q_1 = .44545/.9091 = .49;\ q_2 = .06365/.9091 = .07;\ q_3 = .4/.9091 = .44$$

$$E_{\mathbb{Q}}[S_0] = [10q_1 + 20q_2 + 30q_3] = 19.5 = S_0/B_0;\ S_0 = 17.7275$$

3. This is a fairly simple example because the bond is riskless. Nevertheless, we separate and solve the differential equation representing the bond's price path as follows:

$$\frac{dB_t}{B_t} = rdt \qquad \int\frac{dB_t}{B_t} = \int rdt \qquad ln\,B_T + k_1 = rT + k_2$$

$$e^{\ln B_T} = e^{rT+K} \qquad B_T = e^{rT}e^{K} \qquad B_T = B_0e^{rT}$$

Since the bond is riskless, its value will certainly exceed *X* at time *T*, and the riskless return is also the discount rate on the bond and exercise money, the call value is:

$$c_T = [B_0 e^{rT} - X]e^{-rT} = B_0 - Xe^{-rT}$$

4. The bond's value at time t=10 is given by $B(10) = 100e^{\left(\int_0^{10} r(t)dt\right)}$:

$$B(10) = 100e^{\left(\int_0^{10}(.05-.001t)dt\right)} = 100e^{\left(.05t-.0005t^2|_0^{10}\right)} = 100e^{.45} = 156.83$$

5. Observe that the given differential is separable:

$$\frac{1}{\sqrt{r}}dr = .02\frac{1}{\sqrt{t}}dt.$$

Integrate both sides of the equation:

$$\int r^{-1/2}dr = .02\int t^{-1/2}dt$$

$$2r^{1/2} = .04t^{1/2} + C$$

Since *r(0) = .0225*, then: $2\sqrt{.0225} = 0 + C$, and so *C = .3*. Solving for *r(t)* results in:

$$r(t) = \left(.02\sqrt{t} + .15\right)^2.$$

The interest rate at time *t = 25* is: *r(25) = .0625*.

6.a. The digital call provides for the option holder to receive \$1, with probability $N(d_2)$. Thus, the future expected value of this digital call is simply $1 \times N(d_2)$. Multiplying this conditional expected future value by its probability $N(d_2)$, and discounting by multiplying by $e^{-rfT}$ yields the following formula for the asset-or-nothing call value:

$$c_0 = e^{-r_f T} N(d_2)$$

To develop a better intuition of this model, carefully read the section in Chapter 7 on Option Pricing: A Heuristic Derivation of the Black Scholes Model. This illustration is quite a bit easier than most illustrations related to the section in Chapter 7.

b. Plug into the formula derived in part a to obtain $d_1 = 1.25$, $d_2 = .904$, $N(d_2) = .817$ and $c_0 = .703$.

7.a. Novation, the process by which a clearinghouse assumes counterparty settlement obligations, functions in much the same way in options markets as in other markets. Essentially, novation transfers credit or counterparty risk from participants in option market transactions to the clearinghouse, frequently owned and backed by an exchange. Each clearing house member, including brokers and certain traders serve as back-up underwriters of counterparty risk. Thus, clearinghouses mitigate counterparty risk by assuming settlement obligations themselves such they become the only source of counterparty risk.

b. Such transactions occur regularly in forward markets. Usually, a major bank or other reputable financial institution steps in to serve as counterparty (including settlement obligations) to each of the two transacting parties such that the financial institution assumes the risk on both sides of the forward market transaction. Similar scenarios also regularly occur in swap and OTC options market transactions. Thus, clearinghouses are not essential to novation; they are simply the natural counterparty for exchange-based transactions.

8. No - In a perfectly efficient market, all securities have zero NPV (by definition of a perfectly efficient market).

# Johns Hopkins University

440.646 Instructor: John Teall
Economics of Derivatives Spring Term 2022

Mid-term Quiz: Practice Version 2; Two Hours, Open Book; Solutions are online

1. A stock that is currently selling for $S_0$ can either increase in price to $uS_0$ at time 1, remain the same at $S_0$ or decline in price to $dS_0$. At time 2, the price could either increase by multiplicative rate $u$ from its time 1 value, remain the same as its time 1 value or decrease to proportion $d$ of its time 1 value. Write all of the elements of the time 1 filtration for this process.

2. Suppose that the price of stock now is $S_0$ = \$1, and its price in 3 months is to be uniformly distributed on the interval [.8, 1.25]. The forward price of a non-dividend-paying stock is often assumed to derive from the riskless rate of return $r$: ($F = S_0e^{rt}$). Find the probability that the stock price will be worth more than its forward price ($F = S_0e^{rt}$) in 3 months if the risk-free rate is 5%.

3. Keeler Company stock is expected to be worth \$10 per share if outcome one is realized and \$5 per share if outcome two is realized. One share of Duffy Company stock is expected to be worth \$2 if outcome one is realized and \$10 if outcome two is realized. The stock of both companies is currently selling for \$6 per share.

   a. What is the risk-free rate of return in this economy?
   b. What is the risk-neutral probability that outcome two will be realized?

4. Suppose that the columns in the following matrix represent payoffs of four securities in a 4-outcome space. Do these four securities payment structures span the 4-outcome space? If not, demonstrate that these four payoff vectors cannot span the 4-dimension state space.

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 5 & 6 & 0 & 0 \\ 9 & 10 & 5 & -2 \\ 6 & -4 & -5 & 2 \end{bmatrix}$$

5. Suppose that the log of the stock price relative follows the zero-drift Brownian motion process below where random variable $z$ is distributed normally with parameters (0,1):

$$ln\frac{S_T}{S_t} = \sigma Z\sqrt{T}$$

with $t < T$. Now, suppose that $S_t = X$. Derive a formula to find the probability $P$ that $S_T > X$. What is the numerical value of this probability?

6. Support the argument that central counterparties (novation) in derivatives markets standardizes credit (or default) risk for all traders.

7. Foreign exchange traders face a number of important risks. What are the primary risks faced by traders of forward contracts in foreign exchange (FX) markets?

8. The following table reflects riskless bond prices, coupon rates and terms to maturity for an economy:

| BOND | PRICE | COUPON RATE | TERM TO MATURITY |
|---|---|---|---|
| A | 1000 | 10% | 1 yr. |
| B | 1000 | 10% | 2 yrs. |
| C | 1000 | 10% | 3 yrs. |
| D | 1000 | 10% | 4 yrs. |

Assuming that the bonds make annual coupon payments at year end, answer the following questions based on the above information:

a. What is the four-year spot rate?
b. What is the two-year forward rate for a loan originated one year from now?
c. If the one-year spot rate were to double, but none of the forward rates were to change, what would the new price be for Bond C? (Note: other spot rates may change also)

## Johns Hopkins University

440.646 Instructor: John Teall

Economics of Derivatives Spring Term 2022

Mid-term Quiz Solutions: Practice Version 2

1. The time 1 filtration is written as follows:

$\Im_1 = \{\emptyset, \{(uS_0, uuS_0), (uS_0, uS_0), (uS_0, udS_0)\}, \{(S_0, uS_0), (S_0, S_0), (S_0, dS_0)\}, \{(dS_0, duS_0), (dS_0, dS_0), (dS_0, ddS_0)\}, \Omega\}$.

2. First notice the forward price will be $1\times e^{.05\times.25} = e^{.0125}$. The density function

$$f(x) = \frac{1}{1.25 - .8} = \frac{1}{.45}$$

implies that:

$$F(X > e^{.0125}) = \int_{e^{.0125}}^{1.25} \frac{1}{.45}dx = \frac{1}{.45}(1.25 - e^{.0125}) = 0.5276, \text{or } 52.76\%$$

3.a. Solve for $r$ as follows:

$10\psi_1 + 5\psi_2 = 6$ $\quad .33333 = \psi_1$

$2\psi_1 + 10\psi_2 = 6$ $\quad .53333 = \psi_2$

$r = 1\div(\psi_1+\psi_2) - 1 = 1\div.86666 - 1 = .153846$ or:

b. $q_2 = \psi_2/(\psi_1 + \psi_2)$; $.61539 = \psi_2 = .53333/(.33333+.53333)$

4. There are many ways to demonstrate this. First, attempt to invert the matrix. It cannot be inverted. This will be apparent when you find in the process that you must divide by zero to do so. This is because any row is a linear combination of other rows. For example, the fourth row equals 2 times the first row plus the second row minus the third row. Furthermore, the columns are linear combinations of one another. For example, the fourth column equals -.4 times the third column.

5. Since a zero-drift Brownian motion process is a martingale, $E[S_T|S_t] = S_t$ with ($t < T$), with a 50% probability that $S_T$ will exceed $S_t$. Alternatively, we can solve as follows:

$$P(S_T > S_t) = P\left(S_t e^{\sigma\sqrt{T}Z} > S_t\right) = P\left(e^{\sigma\sqrt{T-t}Z} > 1\right) = P(Z > 0) = N(0) = .5$$

The key intuition here draws from the premises that $S_t = X$ and that there is zero drift. Without drift, the expected value of a Brownian motion process is its prior value since Brownian motion is a martingale.

6. The central counterparty, often a clearinghouse, is the counterparty for all trades in a given venue (e.g., exchange). Thus, individual trader credit risk is irrelevant to other traders since only the credit risk of the central counterparty and all of its backers, usually all members, is relevant. Thus, credit risk is the same (standardized) for all traders in the venue.

7. Traders of forward contracts, including those in foreign exchange (FX) markets face a number of risks. Among these are:

1. Rate risk:  Exchange rates may change in directions opposite to those anticipated by participants.
2. Credit risk: the other party to the contract may default by not delivering the currency specified in the contract. In many instances, an intermediary such as a large reputable commercial bank may act to ensure that one or both contracting parties will honor their agreements. Dealer and counterparty reputation is key.
3. Liquidity risk: The market participant may have difficulty obtaining the currency she must deliver; he may be "stuck" with a currency that will be difficult to sell. Again, a number of intermediaries may improve liquidity by trading and making markets for various currencies.
4. Trading system risk: The trading platform, ECN, exchange and other communication systems are all subject to malfunction or failure.

8. a. Solve first for the four discrete discount functions: $d_1$=.909, then $d_2$=.826, then $d_3$=.751 and finally $d_4$=.683. Now, we can solve for all four of the relevant spot rates: $r_{0,4} = .10$ (all spot rates in this example equal .10 since coupons are all 10% and all bonds are priced at par ($1,000).

b. $r_{1,3} = [(1+r_{0,3})^3 \div (1+r_{0,1})]^{.5} = .10$; This is simple because all spot rates are 10%.

c. Do not use duration - it is not a good approximation here. The new $r_{0,1}$ equals .20 rather than .10. Each of the 4 discount functions will change as well. But each of the forward rates are still .10 (we know this from 4.a. since all old spot rates were .10 and no forward rates changed). Find the new spot rates as follows:
$r_{0,1} = .20$
$r_{0,2} = ((1+.2)\times(1+.1))$^.5 - 1 = .148913
$r_{0,3} = ((1+.2)\times(1+.1)\times(1+.1))$^.333333333 - 1 = .132371
$r_{0,4} = ((1+.2)\times(1+.1)\times(1+.1)\times(1+.1))$^.25 - 1 = .12419
Now find discount functions or d's and discount each of the bond's cash flows accordingly: $d_1$=.833, $d_2$=.757, then $d_3$=.689 and finally $d_4$=.626. Therefore, we discount cash flows on bond C to find that $P_C$ = 100*.833 + 100*.757 + 100*.689 + 1,100* .626 = $P_C$ = 916.5.

**Johns Hopkins University**

440.646 Instructor: John Teall

Economics of Derivatives Spring Term 2022

Final Exam: Practice Version 1

1. A binomial option pricing model prices an option based on two potential outcomes. However, suppose that a particular market consists of three securities: a riskless asset paying 10%, shares of common stock and calls written on these shares with an exercise price of 8. This market exists in a single time period framework with three rather than two potential outcomes. The stock, which is currently selling for 10 will either decrease to 5 next year, remain unchanged at 10 or increase to 15. Physical (or real) outcome probabilities are unknown. In this market, can a "trinomial" option pricing model based on construction of hedge portfolios be derived to value this option? If so, how? If not, why not?

2. A stock currently selling for \$40 in a single time period binomial environment has a one-year call option trading on it with an exercise price equal to \$40. The current Treasury bill rate equals .10 and the probability of an upward price movement is projected to be .52. What are the projected multiplicative upward and downward price movements? Your answer should be within .05.

3. The following Itô process describes the price of a given stock:

$$dS_t = .1(S_t, t) + .5(S_t, t)dz$$

a. What is the solution to this stochastic differential equation?

b. Suppose that the stochastic differential equation above applies for a one-year holding period. What are the expected value and variance of the log of price relative for this stock over a one-month period?

4. Estimate the implied Black-Scholes volatility (standard deviation) for the following call: $c_0$ = 10, $S_0$ = 25, $T$ = .5, $X$ = 35 and $r_f$ = .10. Your answer should be within .01.

5. An investor purchases a one-year American call on a stock that currently sells for \$75 a share with $\sigma$ = .15. The stock is expected to go ex-dividend in 3 months with a dividend of \$4. The exercise price of the call is \$85. The riskless return rate is 4%.

a. Based on Black's Pseudo-American call model, what would be the price of the call?

b. Based on the Roll-Geske-Whaley model, what would be the price of the call?

6. What does an option *vega* measure? Suppose that an option speculator has an option portfolio on a single stock. The speculator believes that the stock risk is going to increase substantially – more so than is reflected in option market prices. Should he increase or decrease his portfolio vega? Should the speculator buy or sell calls to change his portfolio vega? Should he buy or sell puts to change his portfolio vega?

7. If $S_t = (20 + .1t)e^{.05Z_t}$, find $dS_t$.

**Johns Hopkins University**

440.646 Instructor: John Teall

Economics of Derivatives Spring Term 2022

Final Exam Solutions: Practice Version 1

1. A "trinomial" option pricing model cannot be derived because a hedge portfolio cannot be constructed in a one time period, three-potential outcome scenario. That is, no value for $\alpha$ satisfies:

$$\alpha uP_0 - c_u = \alpha P_0 - c_{no\ change} = \alpha dP_0 - c_d$$

$$\alpha 15 - 7 = \alpha 10 - 2 = \alpha 5 - 0$$

Similarly, markets are not complete. There do not exist three priced securities (only two) whose payoff vectors span the 3-dimensional payoff space.

2. Solve the following for $u$:

$$p = \frac{e^{r_f T} - d}{u - d} = .52 = \frac{e^{.1\times 1} - 1/u}{u - 1/u}$$

Thus, the multiplicative upward and downward movements for the stock are projected to be 1.5 and .6667, respectively. Substitute 1.5 for $u$ to verify that $p$ = .52 or solve algebraically for $u$.

3.a. $S_T = S_0 e^{\left[\left(.1 - \frac{.5^2}{2}\right)T + .5z_t\right]}$

b. $E\left[\ln\frac{S_{1/12}}{S_0}\right] = \mu T - \frac{1}{2}\sigma^2 T = \left(.1 - \frac{.5^2}{2}\right)\cdot 1/12 = -.0020833$

4. Try an initial guess for standard deviation equal to, for example, .5. It doesn't matter much what your initial trial estimate is. This .5 estimate results in a call value equal to 1.93. This call value is much too small, so increase the standard deviation estimate. Try a much larger estimate, say $\sigma$ = 2. This estimate results in a call value that is a little large, but reasonably close. So, we try a smaller estimate, iterating until our trial value seems close enough. Ultimately, we arrive at an estimate within .01 of the correct standard deviation of 1.798.

5.a. First, we check to see if the dividend is greater than the time value associated with the exercise money:

$$4 > 85\left(1 - e^{-.04(1-.25)}\right) = 2.512.$$

Since the dividend is greater, the call should be exercised early. Next, we calculate the call value assuming it trades just before the ex-dividend date:

$$d_1 = \frac{ln(75/85) + \left(.04 + \frac{1}{2}.15^2\right)(.25)}{.15\sqrt{.25}} = -1.498009,\ d_2 = -1.498009 - .15\sqrt{.25} = -1.573009$$

$$c_0^* = 75N(-1.498009) - 85e^{-.04\times .25}N(-1.573009) = .16088$$

Now, we calculate the call assuming the option is held until it expires:

$$d_1 = \frac{ln\left(\frac{75 - 4e^{-.04\times .25}}{85}\right) + \left(.04 + \frac{1}{2}.15^2\right)(1)}{.15\sqrt{1}} = -.8544064,\ d_2 = -.8544064 - .15\sqrt{1} = -1.0044064$$

$$c_0^{**} = (75 - 4e^{-.04\times .25})N(-.8544064) - 85e^{-.04\times 1}N(-1.0044064) = 1.08502$$

The value of the call is the maximum of the two call values:

$$c_0 = Max(.16088, 1.08502) = \$1.08502.$$

b. First, we need to find the critical stock price $S_{tD}^*$ so that

$$S_{tD}^* N\left(\frac{ln(S_{tD}^*/85) + \left(.04 + \frac{1}{2}.15^2\right)(1-.25)}{.15\sqrt{1-.25}}\right) - 85e^{-.04\times(1-.25)}N\left(\frac{ln\left(\frac{S_{tD}^*}{85}\right) + \left(.04 - \frac{1}{2}.15^2\right)(1-.25)}{.15\sqrt{1-.25}}\right) = S_{tD}^* + 4 - 85.$$

Using an approximation method such as the method of bisection, one finds that $S_{tD}^* = 90.86189$. As long as the stock price on the ex-dividend date $t_D = .25$ exceeds \$90.86189, then the call will be exercised early. To determine the value of the call:

$$d_1 = \frac{ln\left(\frac{75-4e^{-.04\times.25}}{85}\right) + \left(.04 + \frac{1}{2}.15^2\right)(1)}{.15\sqrt{1}} = -.8544064 \ , d_2 = -.8544064 - .15\sqrt{1} = -1.0044064 \ ,$$

$$y_1 = \frac{ln\left(\frac{75-4e^{-.04\times.25}}{90.86189}\right) + \left(.04 + \frac{1}{2}.15^2\right)(.25)}{.15\sqrt{.25}} = -3.110505 \ , y_2 = -3.110505 - .15\sqrt{.25} = -3.185505.$$

The value of the call is:

$$c_0 = (75 - 4e^{-.04\times.25})N(-3.110505) + (75 - 4e^{-.04\times.25})M(-.8544064, 3.110505, -\sqrt{\frac{.25}{1}})$$

$$-85e^{-.04\times1}M\left(-1.0044064, 3.185505, -\sqrt{\frac{.25}{1}}\right) - (85-4)e^{-.04\times.25}N(-3.185505)$$

$$= 1.08526.$$

$$d_1 = \frac{\ln\left(\frac{40}{37}\right) + \left(.06 - .04 + \frac{1}{2}.12^2\right)(.5)}{.12\sqrt{.5}} = 1.079063 \ , d_2 = 1.079063 - .12\sqrt{.5} = .9942103$$

$$c_0 = 40e^{-.04\times.5}N(1.079063) - 37e^{-.06\times.5}N(.9942103) = 3.54858.$$

$$p_0 = 3.54858 + 37e^{-.06\times.5} - 40e^{-.04\times.5} = .247118.$$

$$d_1 = \frac{\ln\left(\frac{.14}{.12}\right) + (.04 - .05 + \frac{1}{2}.2^2)(.25)}{.2\sqrt{.25}} = 1.566507, d_2 = 1.566507 - .2\sqrt{.25} = 1.466507$$

$$c_0 = .14e^{-.05\times.25}N(1.566507) - .12e^{-.04\times.25}N(1.466507) = .019816.$$

6. Vega measures the sensitivity of the option price to the underlying stock's standard deviation of returns. If an investor believed that stock risk was going to increase, he should increase the vega of his option portfolio. This means that he should buy both calls and puts to increase his portfolio vega.

7. By the general power rule:

$$dS_t = .1e^{.05Z_t}dt + (20 + .1t)d(e^{.05Z_t}) + .1d(e^{.05Z_t})dt.$$

By Itô's Lemma:

$$d(e^{.05Z_t}) = .05e^{.05Z_t}dZ_t + .00125e^{.05Z_t}dt.$$

Since $dZ_t dt$ and $(dt)^2$ are negligible, then:

$$dS_t = .1e^{.05Z_t}dt + (20 + .1t)(.05e^{.05Z_t}dZ_t + .00125e^{.05Z_t}dt)$$
$$= (.125 + .000125t)e^{.05Z_t}dt + (1 + .005t)e^{Z_t}dZ_t.$$

**Johns Hopkins University**

440.646 Instructor: John Teall
Economics of Derivatives Spring Term 2022

Final Exam: Practice Version 2

1. Write the solution to the following stochastic differential equation:

$$dS_t = (\mu + at)S_t dt + \sigma S_t dZ_t$$

with initial value $S(0) = S_0$.

2. Smedley Company stock is currently selling for $40 per share. Its historical variance of returns is .25, compared to the historical market variance of .10. The current one-year treasury bill rate is 5%. Assume that all of the standard Black-Scholes Option Pricing Model assumptions hold.
   a. What is the current value of a put on this stock if it has an exercise price of $35 and expires in one year?
   b. What is the probability implied by problem inputs that the value of the stock will be less than $30 in one year?

3. A share of ACME Corporation currently sells for $45 with a projected annual return standard deviation of .3. A compound call on a call has an exercise price of $4 expiring in 3 months. The underlying call has an exercise price of $40 expiring in 1 year. The riskless return rate is 4%. What is the value of the compound call?

4. Currency options with the following terms are being offered on U.S. dollars (*USD*) denominated in Latvian lat (*LVL*):

| | | | | | |
|---|---|---|---|---|---|
| $T$ | = | .5 years | $r(f)$ | = | .1 |
| $r(d)$ | = | .1 | $S_0$ | = | LVL1.8/USD |
| $\sigma$ | = | .4 | $X$ | = | LVL1.8/USD |

   a. Evaluate the European call.
   b. Evaluate the European put.
   c. What is the probability that the value of the lat (LVL) will exceed $.5556 in six months?

5. Consider an American call option on a non-dividend-paying stock where the stock price is $10, strike price is $7, the stock volatility ($\sigma$) is 10%, and the time to maturity(T) is 6 months. Suppose risk-free interest rate is 4% per annum.
   a. What is the price of this call using a two-period tree? Assume $u$=1.5 and $d$=.5.
   b. What is the price of this call using the Black-Scholes Option Pricing Model?
   c. Now let $u = e^{\sigma\sqrt{t}}, d = \frac{1}{u},\ q = \frac{e^{rt}-d}{u-d},\ and\ t = \frac{T}{n}$, What is the price of the American call option using a two-step tree? (This is called the Cox, Ross & Rubinstein (CRR) method)
   d. What would you conclude about CRR relative to Black-Scholes for an example like this?

6. Show that the solutions for a European call for its price:

$$c_o = S_0 N(d_1) - Xe^{-rT} N(d_2)$$

are equivalent regardless of whether we define:

a.) $d_1 = \frac{ln\left(\frac{S_0}{X}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ *and* $d_2 = \frac{ln\left(\frac{S_0}{X}\right)+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$, *or*

b.) $d_1 = \frac{ln\left(\frac{S_0}{X}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ *and* $d_2 = d_1 - \sigma\sqrt{T}$.

7. Currency options with the following terms are being offered on Swiss Francs (CHF; the home currency is the USD):

| | | |
|---|---|---|
| T | = | .5 years |
| r(f) | = | .1 |
| r(d) | = | .1 |
| $S_0$ | = | CHF1.8/USD |
| σ | = | .4 |
| X | = | CHF1.8/USD |

a. Evaluate a call for this currency European currency option series in terms of USD.
b. Evaluate a put for this currency European currency option series in terms of USD.
c. What is the probability that the value of the franc will exceed USD0.555 in six months?

**Johns Hopkins University**

440.646 Instructor: John Teall

Economics of Derivatives Spring Term 2022

Final Exam Solutions: Practice Version 2

1. First, write the stochastic differential equation in the form:

$$\frac{dS_t}{S_t} = (\mu + at)dt + \sigma dZ_t.$$

By Itô's Lemma, note that:

$$d(lnS_t) = \frac{1}{S_t}dS_t - \frac{1}{2S_t^2}(dS_t)^2 = \frac{1}{S_t}dS_t - \frac{\sigma^2}{2}dt.$$

Thus:

$$d(lnS_t) = \left(\mu - \frac{\sigma^2}{2} + at\right)dt + \sigma dZ_t.$$

Integrating both sides of this equation from 0 to T:

$$ln\left(\frac{S_T}{S_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right)T + \frac{1}{2}aT^2 + \sigma Z_T.$$

Solving for $S_T$:

$$S_T = S_0 e^{\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \frac{1}{2}aT^2 + \sigma Z_T\right]}.$$

2. a. $d_1$= .6172; $d_2$= .1178; $N(d_1)$ = .7314; $N(d_2)$ = .5469

$c_0$= 11.058; with put-call parity: p0 = 4.35

b. Use X=30; $d_1$= .925; $d_2$= .4245; $N(d_2)$ = .6644

$1-N(d_2)$ = .3356

3. The critical underlying stock price $S_{T1}^*$ must satisfy the equation:

$$c_{u,T1} = S_{T1}^* N\left(\frac{ln(S_{T1}^*/40) + \left(.04 + \frac{1}{2}.3^2\right)(1 - .25)}{.3\sqrt{1 - .25}}\right) - 40e^{-.04(1-.25)} N\left(\frac{ln\left(\frac{S_{T1}^*}{40}\right) + \left(.04 - \frac{1}{2}.3^2\right)(1 - .25)}{.3\sqrt{1 - .25}}\right) = 4.$$

Using a desired approximation method such as the method of bisection, one finds that

$$S_{T1}^* = 38.79585$$

Next, we calculate:

$$d_1 = \frac{ln(45/38.79585) + \left(.04 + \frac{1}{2}.3^2\right)(.25)}{.3\sqrt{.25}} = 1.1306614, d_2 = 1.130661 - .3\sqrt{.25} = .9806614$$

$$y_1 = \frac{ln(45/40) + \left(.04 + \frac{1}{2}.3^2\right)(1)}{.3\sqrt{1}} = .67594345, y_2 = .67594345 - .3\sqrt{1} = .37594345$$

We can now directly calculate the value of the compound call:

$$c_{0,call} = 45M\left(1.1306614, .67594345, \sqrt{\frac{.25}{1}}\right) - 40e^{-.04\times 1}M\left(.9806614, .37594345, \sqrt{\frac{.25}{1}}\right) - 4\times e^{-.04\times .25}N(.9806614) = 5.17512.$$

4.a. $d_1 = .1414$; $d_2 = -.1414$; $N(d_1) = .5562$; $N(d_2) = .4438$; $c_0 = .08558$

b. $p_0 = Xe^{-rt} + c_0 - S_0 = .04656$

c. $1 - N(d_2) = .5562$

5.a. We solve for the value of the call using the binomial model as follows:

$$q = \frac{1+r-d}{u-d} = \frac{1+.01-.5}{1.5-.5} = \frac{.51}{1} = .51$$
$$uuS_0 = 1.5\times 1.5\times 10 = 22.5$$
$$duS_0 = udS_0 = 1.5\times .5\times 10 = 7.5$$
$$ddS_0 = .5\times .5\times 10 = 2.5$$
$$c_0 = \frac{\sum_{j=0}^{T}\frac{T!}{j!(T-j)!}q^{\,j}(1-q)^{T-j}MAX[u^j d^{T-j}S_0 - X, 0]}{(1+r)^T}$$
$$c_0 = \frac{\frac{2!}{2!0!}(.51)^2(.49)^0 Max(22.5-7,0) + \frac{2!}{1!1!}(.51)^1(.49)^1 Max(7.5-7,0) + \frac{2!}{0!2!}(.51)^0(.49)^2 Max(2.5-7,0)}{(1+.01)^2}$$
$$= \frac{.2601(15.5) + 2(.51)(.49)(.5) + 0}{1.01^2} = \frac{4.53135}{1.0201} = 4.442$$

b. $$d_2 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{10}{7}\right) + \left(.04 - \frac{1}{2}(.1^2)\right).5}{.1(\sqrt{.5})} = 5.292$$
$$d_1 = d_2 + \sigma\sqrt{T} = 5.292 + .1\left(\sqrt{.5}\right) = 5.363$$

From the z-table, $N(d_1) = N(5.363) = 1$ ; $N(d_2) = N(5.292) = 1$

$$c_0 = S_0N(d_1) - Xe^{-rt}N(d_2) = 10\times 1 - 7\times e^{-.04\times .5}\times 1 = 3.139$$

c. $u = e^{\sigma\sqrt{t}} = e^{.1(\sqrt{.25})} = 1.0513$

$$d = \frac{1}{u} = \frac{1}{1.0513} = 0.9512$$
$$q = \frac{e^{rt} - d}{u-d} = \frac{e^{.04\times .25} - .9512}{1.0513 - .9512} = .5879$$

$uS_0 = 1.0513\times 10 = 10.513$ $\quad uuS_0 = 1.0513\times 1.0513\times 10 = 11.052$

$dS_0 = .9512\times 10 = 9.512$ $\quad ddS_0 = .9512\times .9512\times 10 = 9.048$

$$udS_0 = duS_0 = 1.0513\times .9512\times 10 = 10$$
$$c_0 = \frac{\sum_{j=0}^{T}\frac{T!}{j!(T-j)!}q^{\,j}(1-q)^{T-j}MAX[u^j d^{T-j}S_0 - X, 0]}{(1+r)^T}$$
$$c_0 = \frac{\frac{2!}{2!0!}(.5879)^2(.4121)^0 Max(11.052-7,0) + \frac{2!}{1!1!}(.5879)^1(.4121)^1 Max(10-7,0) + \frac{2!}{0!2!}(.5879)^0(.4121)^2 Max(9.048-7,0)}{(1+.01)^2}$$
$$= \frac{.3456(4.052) + 2(.5879)(.4121)(3) + (.4121^2)(2.048)}{1.01^2} = \frac{3.2018}{1.0201} = 3.139$$

e. Calculating the price using the CRR binomial tree model closely approximates the price using the Black-Scholes model.

6. If we define $d_1$ and $d_2$ as in part a, then:

$$d_2 = \frac{ln\left(\frac{S_0}{X}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

We can rewrite this as:

$$d_2 = \frac{ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)T - \sigma^2 T}{\sigma\sqrt{T}} = \frac{ln\left(\frac{S_0}{X}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} - \sigma\sqrt{T}$$

$$= d_1 - \sigma\sqrt{T}.$$

This is the same definition $d_2$ in part b. This matters for two reasons. First, different books and manuals define d2 as in part a, and others as in part b. Second, it is useful to calculate $d_2$ independently of $d_1$ for the purpose of obtaining implied probabilities that the underlying security price will be within certain ranges.

7.a. Based on the Garman- Köhlagen Model for currency calls, we value the .5-year call on CHF as follows:

$$d_1 = \frac{ln\left(\frac{1.8}{1.8}\right) + \left(.1 - .1 + \frac{1}{2}.4^2\right) \times .5}{.4\sqrt{.5}} = .1414$$

$d_2 = .1414 - .4\sqrt{.5} = -.1414$

$N(d_1) = .5562$; $N(d_2) = .4438$

$c_0 = \mathbf{.193}$

b. Based on put-call parity for currency options, $p_0 = .00147 + .2 \times e^{-.02\times 2} - .15e^{-.06\times 2} = \mathbf{.193}$, the same value as for the call, but only because the exercise price equals the exchange rate and the two interest rates are the same.

c. $1-N(d_2) = .5562$