Chapter 2: Discrete Time and State Models

A. Brief Review of Time Value

Financial markets enable individuals and institutions to accomplish two important things: allocate capital so that it is placed in its most valuable use over time and allocate risk so that those who are least able to bear it pass it on to those with greater risk-bearing capacity. This chapter is concerned first with allocating capital over time and later with the allocation of risk in financial markets.

Cash flows realized at the present time are worth more to investors than cash flows realized later. The present value concept offers a means to express the value of a one dollar (or other currency unit) cash flow in terms of current cash flows:[1]

$$PV = (1 + r_{0,n})^{-n} = d_n \tag{1}$$

where PV is the present value of \$1 paid n equal time periods from now (time 0) with a discount rate (or interest rate or yield) of $r_{0,n}$ (sometimes shortened to r) prevailing from period 0 to period n. The term d_n can be characterized as the value or current price of \$1 to be received at time n. The discount function PV, which expresses the present value of \$1 to be received at time n, can be written as d_n. However, if interest compounds m times per period, present value is computed as follows:

$$PV = \left(1 + \frac{r_{0,n}}{m}\right)^{-mn} \tag{2}$$

As $m \to \infty$, compounding occurs continuously, and the present value function becomes:

$$PV = e^{-n\ {}_{0,n}} \tag{3}$$

Yield Curves

Thus far, we have assumed that discount and interest rates are equal for all periods, along with all bond yields. This means that the *yield curve*, which depicts the yield to maturity of zero-coupon bonds with respect to their terms to maturity, is flat.[2] But, interest rates do change over time, sometimes very predictably, and long-term rates frequently exceed short term rates. Now, we will allow for yields to vary over terms to maturity, and express long term interest rates as functions of short-term rates. We will distinguish between yields or rates on instruments originating at time zero or now (spot rates, on instruments originating now) and yields or rates on instruments originating in the future (forward rates, on instruments to be originated in the future at rates locked in now). More specifically, we will argue that long term interest rates are a geometric mean of a series of short-term spot rates and forward rates. The compounding effect of interest rates leads to long term rates being calculated based on geometric rather than arithmetic means. More specifically, the long-term spot rate will be expressed as a geometric mean of short term spot and forward interest rates.

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[1] Much of the material in this chapter was adapted from Knopf and Teall [2015].

[2] A zero coupon bond, also known as a pure discount bond or strip, makes no explicit interest payments, but is purchased at a discount from its face or maturity value.

The Term Structure of Interest Rates

The *Term Structure of Interest Rates* is concerned with how yields and interest rates vary with respect to dates of maturity. *The Pure Expectations Theory* defines the relationship between long and short term interest rates as follows, where $r_{0,n}$ is the rate on an instrument originated at time 0 and repaid at time n:

(12) $$r_{0,n} = \sqrt[n]{\prod_{t=1}^{n}(1+r_{t-1,t})} - 1$$

Illustration: Term Structure

Consider an example where the one year spot rate $r_{0,1}$ is 2%. Investors are expecting that the one year spot rate one year from now will increase to 3% such that the one year forward rate $r_{1,2}$ on loans originated in one year is 3%. Further suppose that investors are expecting that the one year spot rate two years from now will increase to 5%; thus, the one year forward rate $r_{2,3}$ on a loan originated in two years is 5%. Based on the pure expectations hypothesis, the three-year spot rate is calculated as follows:[3]

$$r_{0,3} = \sqrt[n]{\prod_{t=1}^{n}(1+r_{t-1,t})} - 1 = \sqrt[3]{(1+r_{0,1})(1+r_{1,2})(1+r_{2,3})} - 1 = \sqrt[3]{(1+.02)(1+.03)(1+.05)} - 1 = .033258$$

B. Arbitrage and No Arbitrage Pricing

As we discussed in Chapter 1, *arbitrage* is the simultaneous purchase and sale of instruments producing identical cash flows. The most easily arbitraged financial instruments are those with guaranteed payments or with payments that are perfectly correlated with other instruments. We will focus on riskless bonds here, though the principles that we discuss can be extended to other securities. The key components of arbitrage are that:

1. Arbitrage is riskless. All cash flows, including transactions prices in the market are known.
2. Arbitrage will never produce a negative cash flow in any time period or under any outcome.

An arbitrage opportunity fulfills the above conditions and produces at least one positive cash flow in at least one period and/or outcome. In a perfect market, arbitrage opportunities do not exist since rational and greedy investors will never price securities such that they produce an arbitrage opportunity for a competitor at their own expense.

No-arbitrage conditions are used to price securities relative to one another such that they do not produce such an arbitrage opportunity. *No-arbitrage pricing* is based on the simple *Law of One Price*, which holds that securities or portfolios producing identical cash flow structures must sell for the same price. No-arbitrage pricing holds that all securities must be priced so that the Law of One Price is not violated. Furthermore, self-financed arbitrage portfolios must produce zero cash flows.

[3] For more details on obtaining spot and forward rates in bon markets, see Teall [2018] or Knopf and Teall [2015]. Knopf and Teall [2015] also provides detail on term structure theories involving uncertain shifts in spot and forward rates.

C. Probability and Risk

Risk has many definitions, sometimes these meanings are inconsistent or even contradictory, but most definitions are somehow related to uncertainty or undesirable outcomes. Risk can refer to uncertainty, volatility, bad outcomes (e.g., bankruptcy), probability of bad outcomes, extent of bad outcomes, certainty of bad outcomes, etc. Measuring risk is frequently complicated by the difficulty in defining risk, but nonetheless is often accomplished with a variety of tools drawn from probability and statistics. Our usage of the term risk in this book will usually be related to uncertainty, and our measurements will usually pertain to uncertainty or volatility.

Sets and Measures

We require a set of rules and grammar to effectively communicate with words.[4] Similarly, rules and syntax are needed to communicate ideas in mathematics and probability. Here, we begin to set forth basic rules and syntax for discussions related to probability.

Sets

A *set* A is a collection of well-defined objects called elements such that any given object x either (but not both) belongs to A ($x \in$ A) or does not belong to A ($x \notin$ A). The *cardinality* of the set is its number of members, which can be either finite or infinite. Set A is a subset of B (A $\subseteq$ B) if and only if every element of A is also an element of B; set A is a proper subset of B (A $\subset$ B) if and only if every element of A is also an element of B but A $\neq$ B. The *union* of sets A and B, A $\cup$ B, is the set of all distinct elements that are either in A or B, or both in A and B. The *intersection* of sets A and B, A $\cap$ B, is the set of all distinct elements they share in common.

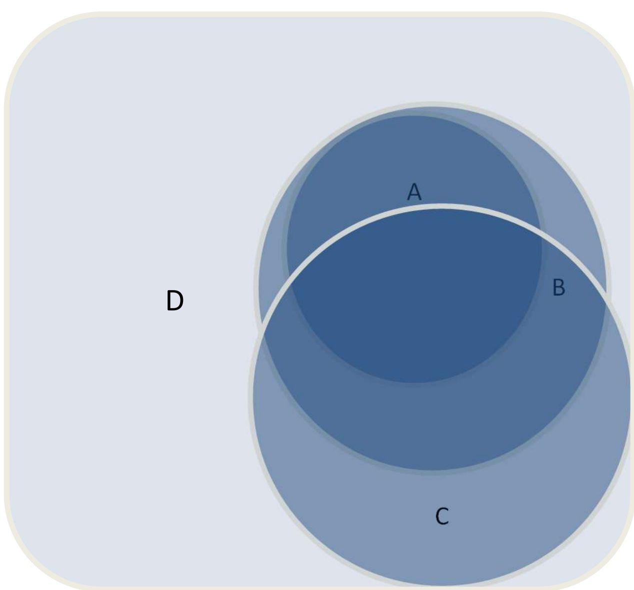

Figure 1: Venn Diagram of Sets and Subsets

[4] Billingsley (1995) provides an excellent formal mathematical introduction to probability theory.

Illustration: Toss of 2 Dice

Consider an experiment based on the total of a single toss of two dice. Let *D* be the set of all possible outcomes, which range from 2 to 12. Let *B* be the set of outcomes between 5 and 10 inclusive. Let *C* be the set of all outcomes not less than 9. Let *A* be the set of odd outcomes between 5 and 9 inclusive. Thus, we have the sets A={5,7,9}, B={5,6,7,8,9,10}, and C={9,10,11,12}. These sets are depicted in Figure 1. We can infer the following:

$$A \subset B \subset D$$

Since {9, 10} contains the only elements in both sets B and C, we can write:

$$B \cap C = \{9, 10\}$$

That is, the intersection of Sets B and C is {9, 10}. One can also check that the union of sets B and C equals:

$$B \cup C = \{5,6,7,8,9,10,11,12\}.$$

Measurable Spaces and Measures

A measure is a function that assigns a real number to subsets of a given space. For example, in 3-dimensional space, the *Lebesgue measure* of any solid shape (the measureable sets in this space) is simply its volume. In this text, we will discuss a particular type of measure called a probability measure. We will provide a definition of a probability measure shortly. As an introduction, two necessary properties of a probability measure are:

1. The measure of any subset of the space is nonnegative.
2. The measure of the entire space is 1.

A measurable space (Ω,Φ) is a collection of *outcomes* ω comprising the set Ω, referred to as the *universal set* or *sample space*, and a set Φ, a specified collection of subsets of Ω. The set Φ, comprising *events* ϕ, forms a *σ-algebra*, which has the following three properties:

1. $\emptyset \in \Phi$ and $\Omega \in \Phi$ where $\emptyset = \{\ \}$ is the empty set (set containing no elements)
2. If $\phi \in \Phi$, then $\phi^c \in \Phi$ (ϕ^c is the complement of ϕ)
3. If $A_i \in \Phi$ for every positive integer *i*, then $\bigcup_{i=1}^{\infty} A_i \in \Phi$.

The σ-algebra contains every countable or finite intersection of sets taken from the σ-algebra.[5] The idea behind the concept of a σ-algebra is to have a sufficient variety of subsets of the sample space, to which we will later assign probabilities. In the case that the sample space Ω is finite or countable, then usually the σ-algebra Φ consists of the power set (set of all possible subsets) of Ω. In the case when Ω is the set of real numbers or an interval of real numbers, then the σ-algebra Φ normally consists of all sets that are generated by countable unions and complements of intervals contained in the sample space Ω. The reason why the σ-algebra is not chosen to be the power set in the case of real numbers is because it is usually impossible to assign

[5] This follows from de Morgan's Law: $\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c\right)^c$.

probabilities the way we would like to every possible subset of the real numbers.

Probability Spaces

A probability space involves assigning or mapping probabilities (numbers between zero and one) to the possible outcomes or events that can occur. A probability space (Ω,Φ, P) consists of a measurable space (Ω, Φ) and a probability measure *P*, to be defined shortly. A *probability space* consists of a sample space Ω, events Φ and their associated probabilities *P*. The probability (measure) *P* is a function that assigns a value to each event ϕ taken from Φ so that the following properties are satisfied:

1. For every event ϕ taken from Φ, $0 \leq P(\phi) \leq 1$.
2. $P(\Omega) = 1$.
3. For any set of pair-wise disjoint (mutually exclusive) events $\{\phi_i\}_{i=1}^{\infty}$, we have $P(\bigcup_{i=1}^{\infty} \phi_i) = \sum_{i=1}^{\infty} P(\phi_i)$.[6]

A probability space can be considered to be the triplet (Ω, Φ, P), where again, Ω is the set of outcomes, Φ is a σ-algebra of subsets of Ω, and P assigns probabilities between 0 and 1 to event sets ϕ in Φ.

Physical and Risk Neutral Probabilities

For financial valuation purposes, it is essential to realize from the outset what is and is not included in the definition for a probability or probability measure. The three properties set forth above essentially say that probabilities associated with events range between 0 and 1, they sum to 1 and that the probability of the union of a set of mutually exclusive events equals the sum of the probabilities for each of the events in the union. Nowhere does this definition of a probability say anything about frequencies or likelihoods of occurrences. This is not part of the general definition that we will use for probability.

However, we often do find it convenient to associate probabilities with frequencies and likelihoods of events. We will refer to these probabilities as physical probabilities. That is, a *physical probability* is a specific type of probability that reflects the frequency or likelihood of an event. While these types of probabilities have a certain intuitive appeal, they are not usually very helpful for financial valuation using no-arbitrage pricing. First, we often have no meaningful methodology to determine or observe these physical probabilities. Different analysts perceive them differently. Worse, expected values derived from them frequently lead to valuations that lead to price inconsistencies (we will refer to this as arbitrage later in the text). That is, the use of physical probabilities simply leads to incorrect market valuations. This inconsistency problem intensifies as investors become increasingly risk averse.

In the next chapter, we will synthesize probabilities that lead to correct and consistent valuations. These synthetic probabilities are known as *risk-neutral probabilities*, which are presumed to be used by risk-neutral investors valuing assets in a consistent manner. Nonetheless, both physical probabilities and risk-neutral probabilities fulfill the conditions set forth above.

[6] The events $\{\phi_i\}_{i=1}^{\infty}$ are said to be *pairwise disjoint* (mutually exclusive) if for any *i≠j,* the intersection $\phi_i \cap \phi_j = \emptyset$, the empty set.

Illustration: Probability Space

Suppose that a stock will either increase (u) or decrease (d) in each of two periods with equal probabilities. The sample space of outcomes is the set $\Omega = \{uu, ud, du, dd\}$ of all potential outcomes or elementary states ω of the sample space ("world") Ω. An event ϕ is a subset of elementary states ω taken from Ω and is an element of the set of events Φ. In this illustration, we choose Φ to be the power set of Ω. If a set Ω has n elements, its power set has 2^n elements ϕ. Thus, in this case the set of events Φ has $2^4 = 16$ events:

∅	{dd}	{ud,du}	{uu, ud,dd}
{uu}	{uu,ud}	{ud,dd}	{uu, du,dd}
{ud}	{uu,du}	{du,dd}	{ud,du,dd}
{du}	{uu,dd}	{uu, ud,du}	{uu, ud,du,dd}

Suppose that physical probabilities associated with each two-period outcome ω (a pair involving u and d) equals .25. Physical probabilities associated with each event in the σ-algebra are then:

P{∅} = 0	P{dd} = .25	P{ud,du} = .5	P{uu, ud,dd} = .75
P{uu} = .25	P{uu,ud} = .5	P{ud,dd} = .5	P{uu, du,dd} = .75
P{ud} = .25	P{uu,du} = .5	P{du,dd} = .5	P{ud,du,dd} = .75
P{du} = .25	P{uu,dd} = .5	P{uu, ud,du} = .75	P{uu, ud,du, dd} = 1

Random Variables

A *random variable X* defined on a probability space (Ω,Φ,P) is a function from the set Ω of outcomes to ℝ (the set of real numbers) or a subset of the real numbers. A random variable X is said to be *discrete* if it can assume at most a countable number of possible values: $x_1, x_2, x_3,\ldots$ For each individual value x_i of the random variable X, we require that the probability $P(X=x_i)$ be defined. See the illustration below. A random variable X is said to be *continuous* if it can assume any value on a continuous range on the real number line. We require that for any interval (a, b) on the real line that the probability $P(a<X<b)$ be defined. We note that the number a can take on the value $-\infty$, and the number b can take on the value ∞.

Illustration: Discrete Random Variables

Returning to the illustration in the subsection above, the random variable $X(\omega)$ could denote the value of the stock given an outcome ω in the sample space. Suppose that the stock price is defined in the following way: $X(uu) = 30$, $X(ud) = 20$, $X(du) = 20$, and $X(dd) = 10$. Thus, $P(X = 30) = P(\{uu\}) = .25$, $P(X = 20) = P(\{ud,du\})=.5$, and $P(X=10) = P(\{dd\})=.25$. For example, $P(X=20)=.5$ means that there is a .5 probability (50% chance) that the stock price will be 20, and this will happen in the event that there is either an increase followed by a decrease (ud) or a decrease followed by an increase (du). This is an example of a discrete random variable since the stock price can only take on discrete values (10, 20, 30). Note that this example fulfills the main requirement for X to be a random variable, since the probability that X equaled each of its potential values 10, 20, and 30, respectively, were defined: $P(X = 10) = .25$, $P(X = 20) = .5$ and $P(X = 30) = .25$.

Conditional Probability

Conditional probability is used to determine the likelihood of a particular event A contingent on the occurrence of another event B. The probability of event A given B is the probability of both A and B divided by the probability of B:

(11) $$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

The expected value of a random variable *V* conditioned on its exceeding some constant *C* is calculated as follows:

(12) $$E[V|V > C] = \frac{\sum_{V_i > C} V_i P(V = V_i)}{\sum_{V_i > C} P(V = V_i)}$$

in the discrete case, and

(13) $$E[V|V > C] = \frac{\int_C^\infty x p(x) dx}{\int_C^\infty p(x) dx}$$

in the continuous case, with $p(x)$ as the density function for *V*.

Illustration: Drawing a Spade

Suppose that we draw a card at random from an ordinary deck of 52 playing cards. What is the probability that a spade is drawn given that the drawn card is black? Assume that spades and clubs are black cards, and that each type comprises 25% of the deck.

The sample space for this draw has 52 outcomes. Let *A* be the event that a spade is drawn, and let B be the event that a black card is drawn. Since $P[A \cap B]=13/52=1/4$ and $P[B]=26/52=1/2$, $P[A|B]=P[A \cap B]/P[B]=(1/4)/(1/2)=1/2$. Observe that the probability of obtaining a spade out of all 52 outcomes in the sample space is 1/4. However, if we are given the additional information that the card was black, then we are effectively limiting our sample space to the black cards, and the probability of obtaining a spade is 1/2.

Bayes Theorem

Bayes Theorem is useful for computing the conditional probability of an event B given event A, when we know the value of the reverse conditional probability; that is, the probability of event A given event B. In its simplest form, Bayes Theorem states that:[7]

(14) $$P[B|A] = \frac{P[A|B]P[B\}}{P[A]}.$$

D. Discrete State Models

In Section A, we worked with riskless securities in single state, multiple period

[7] Bayes Theorem can be generalized to multiple events. Suppose that events $B_1, B_2, \ldots, B_n$ form a partition of the sample space *S*, which means that the events are pairwise disjoint and their union is the sample space S. If A is an event then:

$$P[B_i|A] = \frac{P[A|B_i]P[B_i]}{\sum_{j=1}^n P[A|B_j]P[B_j]}.$$

frameworks. Now, we will examine multiple state risky markets in a single period.

Outcomes, Payoffs and Pure Securities

Investors can value securities by valuing them as functions of the known prices of other securities which have already been valued. This process can involve determining the vector space $\mathbb{R}^n$, valuing *n* "control" securities with linearly independent payoff vectors, and pricing the payoff vectors of the previously unpriced securities based on linear combinations of prices from the "control" securities. We will initially assume the following for our valuations:

1. There exist *n* potential *states of nature* (prices) in a one-time period framework.
2. Each security will have exactly one payoff resulting from each potential state of nature.
3. Only one state of nature will occur at the end of the period (states are mutually exclusive) and which state occurs is ex-ante unknown.
4. Each investor's utility or satisfaction is a function only of his level of wealth; the state of nature that is realized is important only to the extent that the investor's wealth is affected (this assumption can often be relaxed).
5. Capital markets are in equilibrium (supply equals demand) for all securities.

Payoff Vectors and Pure Securities

Suppose that in a given economy has *n* potential states of nature and there exists a security *x* with a known payoff vector:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Vector **x** defines every potential payoff for security *x* in this *n*-state world. We will value this security based on known values of other securities existing in this three-state economy.

The first step in the evaluation procedure is to decompose the security into an imaginary portfolio of *pure securities*. Define a pure security (also known as an elementary, primitive or Arrow-Debreu security) to be an investment that pays $1 if and only if a particular outcome or state of nature is realized and nothing otherwise. Thus, the payoff vector for a given pure security *i* in an *n*- potential outcome economy will comprise *n* elements:

1. The i^{th} element will equal 1
2. All other elements will be zero.

The following is the payoff vector of pure security 3 in an *n*-outcome economy:

$$\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Each security with a payoff vector in this n-dimensional payoff space can be replicated with the n pure securities from this space.

Payoff Vectors and Pure Securities: Illustration

Suppose that an economy has 3 potential states of nature and there is a security x with the following payoff vector:

$$\mathbf{x} = \begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix}$$

Vector **x** defines every potential payoff in this three-state world. Notice that by expressing a "real" security payoff vector in terms of the pure securities, it tells us immediately the future value of the security given any particular outcome. In this example, security x will pay 8, 4, or 1, respectively, if and only if outcome one, two, or three, respectively occurs. We will value this security based on known values of other securities existing in this 3-state economy.

The first step in the evaluation procedure is to decompose the security into a portfolio of pure securities. For example, the payoff vector for pure security 2 in a three-potential-outcome world is given as follows:

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Pure security 2 will pay 1 if outcome 2 is realized; otherwise, it will pay zero. Payoff vectors for three pure securities will span the vector space for a three-outcome economy:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The 3-dimensional vector space is spanned by three vectors if any vector in that space can be defined in terms of a linear combination of those three vectors. The payoff vector for any security existing in this three-outcome world is a linear combination of the payoff vectors of the three pure securities spanning the three- element vector space. For example, security **x** is a linear combination of pure securities 1, 2 and 3:

$$\begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix} = 8 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{x} = 8 \times \mathbf{e}_1 + 4 \times \mathbf{e}_2 + 1 \times \mathbf{e}_3$$

We are able to evaluate security x easily if we know values of the three pure securities. Suppose just for the moment that the value of pure security 1 is $\psi_1 = .362$, which suggests that an investor is willing to pay .362 for a security that pays 1 if and only if outcome one is realized.

Furthermore, suppose that pure security 2 has a value of .523 and pure security 3 has a value of .015. The value of security *x* is determined as follows: $S_x = 8\,\psi_1 + 4\,\psi_2 + 1\,\psi_3 = (8 \times .362) + (4 \times .523) + (1 \times .015) = 5$. Next, we discuss how we value pure securities.

Spanning and Complete Markets

The *n*-dimensional payoff space for a market is *spanned* when the payoff vectors for a set or subset of *n* securities is linearly independent. Complete capital markets exist when these *n* securities have known prices. In *complete capital markets*, the value of any other security, real or synthetic, with a payoff vector in this *n*-state space is calculated based on values of the original *n* securities. With the known prices of *n* traded securities with linearly independent payoff vectors, we can determine values of each of *n* pure securities.

Spanning and Complete Markets: Illustration

A complete capital market in a *3*-potential outcome economy occurs when a set of *3* securities with known prices spans the state space. If the values of each of these *3* securities are known, the value of any other security in this *3*-potential state world can be determined based on values of the original *3* securities. Suppose, for example, securities *x*, *y* and *f* exist in a 3-potential outcome world and have the following payoff vectors:

$$\mathbf{x} = \begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 2 \\ 10 \\ 3 \end{bmatrix} \quad \mathbf{f} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

This set of three payoff vectors is linearly independent. Thus, **x**, **y** and **f** span the 3-dimensional vector space and, if *x, y* and *f* are priced, there exists a complete capital market in this economy. The payoff vector for every other security in this economy is a linear combination of the payoff vectors for securities *x, y* and *f*. This implies that we can decompose the payoff vectors of securities *x, y* and *f* into the following component pure security payoff vectors:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where security *x* replicates a portfolio comprising 8 of pure security 1, 4 of pure security 2 and 1 of pure security 3. Similarly, security y replicates a portfolio of 2 of pure security 1, 10 of pure security 2, and so on. Suppose that security *x* has a market value of 5, security y has a market value of 6 and *f* has a value of .9. In the absence of arbitrage opportunities, portfolios of pure securities replicating the cash flow structures of *x*, *y* and *f* should also have values of 5, 6 and .9:[8]

(21)
$$\underset{\mathbf{S}}{\begin{bmatrix} 5 \\ 6 \\ .9 \end{bmatrix}} = \underset{\mathbf{CF}}{\begin{bmatrix} 8 & 4 & 1 \\ 2 & 10 & 3 \\ 1 & 1 & 1 \end{bmatrix}} \underset{\boldsymbol{\psi}}{\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}}$$

More generally, suppose we are given *n* priced securities, with future cash flows given by

[8] Upper case **S** is used to denote this vector of security prices so as to not confuse with a lower case **s** used for states.

CF and prices given by vector **S**. Where the set of security payoff vectors is linearly independent and spans the n-outcome payoff space, we can find the n pure security prices, expressed by the vector ψ, by solving the system:

(22) $$\boldsymbol{S} = \mathbf{CF} \times \psi.$$

The solution is:

(23) $$\psi = \mathbf{CF}^{-1}\mathbf{S}.$$

There is a very close connection between pricing bonds based on payoffs in the previous section and pricing securities in this section based on pure security payoffs. Observe that equation (22) is analogous to equation (14), and equation (23) is analogous to equation (15).
For our example in (21), we obtain $\psi_1 = .362$, $\psi_2 = .523$,and $\psi_3 = .015$.

Now, consider a fourth security z with the following payoff structure:

$$\mathbf{z} = \begin{bmatrix} 20 \\ 8 \\ 6 \end{bmatrix}$$

The price $S_{z,0}$ of market security z is determined from the prices of our three pure securities:

$$\begin{array}{cccc} S_{z,0} = & \begin{bmatrix} .362 & .523 & .015 \end{bmatrix} & \begin{bmatrix} 20 \\ 8 \\ 6 \end{bmatrix} & = 11.508 \\ S_{z,0} = & \psi^{\mathrm{T}} & \mathbf{cf_z} & \end{array}$$

Arbitrage and No Arbitrage Revisited

As we discussed earlier, arbitrage is the simultaneous and riskless purchase and sale of instruments producing identical cash flows that produces non-negative cash flows in every state of nature. We will use no-arbitrage conditions to price securities relative to one another such that they do not produce such an arbitrage opportunity. Consider a frictionless market trading n securities i with payoffs $CF_{i,j}$ over n possible future states of nature j (notice that our cash flow and *kernel* matrices may also include time zero cash flows, security prices). Let γ_i represent the investment commitment made to a given investment i; that is, γ_i is the number of units that could be held by an investor. A no-arbitrage (arbitrage-free) market exists where for each and every possible portfolio strategy γ:

$$\begin{array}{cccc} \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{bmatrix} & \begin{bmatrix} CF_{1,0} & CF_{1,1} & CF_{1,2} & \cdots & CF_{1,n} \\ CF_{2,0} & CF_{2,1} & CF_{2,2} & \cdots & CF_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CF_{n,0} & CF_{n,1} & CF_{n,2} & \cdots & CF_{n,n} \end{bmatrix} & \begin{bmatrix} 1 \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix} & = 0 \\ \gamma^{\mathbf{T}} \times & \widehat{CF} & \times\, \hat{\boldsymbol{\psi}} & = 0 \end{array}$$

Observe that this equation is analogous to equation (16) for bonds. Time zero cash flows will be negative (positive) if this is when securities are purchased (short-sold). A non-zero value for this product implies an arbitrage opportunity, either through purchases, short sales or some combination thereof. All securities in the n-state payoff space can be priced in this no-arbitrage market. Thus, in a no-arbitrage market, each security k outside this n-security set ($k \notin \{1, 2, ..., n\}$), but whose payoff vector remains in the n-state payoff space can be priced as a linear combination of the n security set. That is, if the cash flow structure of security k can be replicated by some specific portfolio γ of securities in subset n from the market:

(24)
$$\begin{bmatrix} CF_{k,1} \\ CF_{k,2} \\ \vdots \\ CF_{k,n} \end{bmatrix} = \begin{bmatrix} CF_{1,1} & CF_{1,2} & \cdots & CF_{1,n} \\ CF_{2,1} & CF_{2,2} & \cdots & CF_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ CF_{n,1} & CF_{n,2} & \cdots & CF_{n,n} \end{bmatrix}^T \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}$$
$$\boldsymbol{cf}_k = \boldsymbol{CF}^T \times \boldsymbol{\gamma}$$

Observe that equation (24) is analogous to equation (17) for bonds. Then the price of that security $k \notin \{1,2,...,n\}$ in an arbitrage-free market can be expressed as a linear combination of the prices of securities numbered from set $\{1, 2, .., n\}$:

(25)
$$S_{k,0} = [\gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_n] \begin{bmatrix} S_{1,0} \\ S_{2,0} \\ \vdots \\ S_{n,0} \end{bmatrix}$$

This equation is analogous to equation (18) to price a bond.

The Pricing Kernel

Here, we define the *pricing kernel* to be vector $\widehat{\boldsymbol{\psi}}$ such that $\widehat{\boldsymbol{CF}} \times \widehat{\boldsymbol{\psi}} = [0]$:

(26)
$$\begin{bmatrix} CF_{1,0} & CF_{1,1} & \cdots & CF_{1,n} \\ CF_{2,0} & CF_{2,1} & \cdots & CF_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ CF_{n,0} & CF_{n,1} & \cdots & CF_{n,n} \end{bmatrix} \begin{bmatrix} 1 \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix} = [0]$$
$$\widehat{\boldsymbol{CF}} \times \widehat{\boldsymbol{\psi}} = [0]$$

If asset purchases (sales) are made at time zero, asset purchase prices are negatives (positives) of the market prices at time 0. The pricing kernel, in effect, provides that the price of a security exactly offsets its future value; security prices are exactly what securities are worth based on pure security prices. All securities with payoff vectors in this state space can be priced with this pricing kernel. Elements of the pricing kernel are pure security prices.

Illustration: Obtaining the Pricing Kernel

Consider the three security illustration (securities x, y and f) in the previous section. When the market is in equilibrium, a pricing kernel for a securities market can be produced. We define the pricing kernel for this market to be vector $\widehat{\boldsymbol{\psi}}$ such that $\widehat{\boldsymbol{CF}} \times \widehat{\boldsymbol{\psi}} = [0]$:

$$\begin{bmatrix} -5 & 8 & 4 & 1 \\ -6 & 2 & 10 & 3 \\ -.9 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\widehat{\boldsymbol{CF}} \quad \times \widehat{\boldsymbol{\psi}} \; = \; [0]$$

As we showed earlier, we can also express this equation in the form of equation (22):

$$\begin{bmatrix} 8 & 4 & 1 \\ 2 & 10 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ .9 \end{bmatrix}.$$

By equation (23), the solution is:

$$\begin{bmatrix} 8 & 4 & 1 \\ 2 & 10 & 3 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 6 \\ .9 \end{bmatrix} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$$

$$\begin{bmatrix} .1346 & -.058 & .0384 \\ .0192 & .1346 & -.423 \\ -.1538 & -.0769 & 1.385 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ .9 \end{bmatrix} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} .362 \\ .523 \\ .015 \end{bmatrix}$$

Thus, $\widehat{\boldsymbol{\psi}}$ = [1, .362, .5223, .015]^T is the pricing kernel for this illustration.

The Equivalent Martingale: Synthetic Probabilities

If our analysis includes a riskless asset in the type of no arbitrage pricing model that we have been using in this section, every security in this no arbitrage market will have will have the same expected rate of return as the riskless bond under certain circumstances. An important feature of this type of pricing model in such a market is that it can be used to define "synthetic," "hedging" or "risk-neutral" probabilities q_i. These risk-neutral probabilities do not exist in any sort of realistic sense, nor are they assumed at the start of the modeling process. Instead, they are inferred from market prices of securities and interest rates. These risk-neutral probabilities are essential in that they can be used to calculate the price of any security in the market so that the no arbitrage nature of the market is maintained. Risk neutral probabilities have the useful feature that they lead to expected values that are consistent with pricing by investors that are risk neutral, leading to the term risk neutral pricing. This is important because it means that we do not need to work with unobservable risk premiums when we value securities; in fact, we do not even need to know anything about any investors' risk preferences.

If payoffs for each of the pure securities $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the result of distinct outcomes, which together account for all possible outcomes (thus forming a sample space), then we can view the pricing model in terms of probabilities. The pure security prices ψ_1, ψ_2, ψ_3 are proportional to the market's assessment of the relative likelihood that the outcomes that we number as 1, 2, and 3, respectively, will occur. Consider the example above concerning securities *x*, *y* and *f*. The pure security prices estimated in this example were estimated to be ψ_1 = .362, ψ_2 = .523 and ψ_3 =.015. This suggests that investors would be willing to pay .362 for a security that pays 1 if and only if Outcome 1 is realized; they would be willing to pay .523 for a security that pays 1 if and only if Outcome 2 is realized, and so on. If we assume that investors are risk neutral, we can infer that they believe that Outcome 2 is more likely to be realized than Outcome 1 and both are more likely than outcome 3. Suppose we create the riskless portfolio with payoff vector $\mathbf{f^T} = [1 \quad 1 \quad 1]^T$; that is, the portfolio that pays off 1 regardless of which of the 3 outcomes occurs. From our example, we were given the price that the market gives to this portfolio:

$$[1 \quad 1 \quad 1]\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\end{bmatrix} = \psi_1 + \psi_2 + \psi_3 = .9.$$

However, the pure security prices add to less than one, implying that investors prefer money sooner rather than later. In this example, and as is usually the case, the ψ_i's do not sum to 1 because of investors' time value for money. Nevertheless, if we accept that ψ_i is proportional to the relative likelihood that outcome *i* will occur in the case of a risk neutral investor, then the probability that outcome *i* occurs is

(27) $$q_i = \frac{\psi_i}{\sum_{j=1}^{n} \psi_j}$$

where ψ_i is the price of pure security *i* and ψ_j is the price of each of *n* pure securities *j*. In our illustration, synthetic probabilities are $q_1 = .4022$, $q_2 = .5811$ and $q_3 = .0167$. These probabilities are referred to as synthetic probabilities because they are constructed from security prices rather than directly from investor assessments of physical probability. As we shall discuss in later chapters, this vector of synthetic probabilities is called the equivalent martingale.

Discount Factors

We saw in our example above that our riskless portfolio $[1 \quad 1 \quad 1]$ was priced at .9. This is because there is a time gap between the time the portfolio is priced and when it is paid off. We can use the information obtained from pure security prices to infer the market's time value of money in addition to synthetic probabilities. In a one-time period economy, the discount function and riskless rate are obtained from pure security prices as follows:

(28) $$d_1 = \frac{1}{1+r} = \sum_{j=1}^{n} \psi_j$$

Time one payoffs for a riskless portfolio of pure securities will be 1 in every state of nature; each pure security in the state space will be included in the riskless portfolio. In our example above with three pure securities $\psi_1 = .362$, $\psi_2 = .523$ and $\psi_3 = .015$, d_1 will equal .9 and the riskless return *r* will equal .111. Normally, one would expect that pure security prices will sum to less than 1 such that $d_1 < 1$. From equations (27) and (28), we see that we can also express each synthetic probability in the form:

(29) $$q_i = \frac{\psi_i}{d_1}.$$

In our illustration, synthetic probabilities are q_1 =.362/.9= .402, q_2 = .523/.9=.581 and q_3 =.015/.9= .017.

The Risk Neutrality Argument

In previous sections, we used complete capital markets and no-arbitrage pricing to demonstrate how cash flow structures are replicated and securities are priced relative to other securities. Such replication and pricing are invariant with respect to investor risk preferences. Pure security prices and synthetic probabilities implicitly reflect risk preferences so that such preferences need not be explicitly input into pricing of other securities. Pure security prices and relative pricing relations are enforced by arbitrage. This is the basis of the risk-neutral valuation models. Risk-neutral valuation means that we are able to price securities such that in the risk neutral probability space (synthetic probability space), risky securities such as stocks will have the same expected return the riskless asset such as a T-bill. This will be illustrated in the next section.

Binomial Option Pricing: One Time Period

Derivative securities are assets whose values are derived from the performance of other securities, indices or rates. Stock options are examples of derivative securities. One type of stock option is a *call*, which grants its owner the right (but not the obligation) to purchase shares of an underlying stock at a specified "exercise" price within a given time period (before the expiration date of the call). The expiration payoff of a call is the maximum of either zero or the difference between the stock price S_T at expiration (at time T) and the exercise price X of the call:

$$c_T = MAX[(S_T - X),0]$$

Consider a one-time-period, two-potential-outcome framework where Company *K* stock current sells for \$50 per share and a riskless \$100 face value T-bill sells for \$90. Suppose Company *K* stock will pay its owner either \$20 or \$80 in one year. A call with an exercise price of \$60 underlies *K* stock shares. This call will be worth either \$0 or \$20 when it expires, based on the value of the underlying stock. The payoff vectors **k** for the stock, the T-bill (**b**) and the call (**c**) are given as follows:

$$\mathbf{k} = \begin{bmatrix} 20 \\ 80 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 100 \\ 100 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 0 \\ 20 \end{bmatrix}$$

The current prices of the stock and T-bill are known to be \$50 and \$90. Since their payoff vectors span the two-outcome space in this two-potential-outcome framework, they form complete capital markets and we can estimate pure security prices as follows:

(30)
$$\underset{\mathbf{v}}{\begin{bmatrix} 50 \\ 90 \end{bmatrix}} \underset{=}{=} \underset{\mathbf{CF}}{\begin{bmatrix} 20 & 80 \\ 100 & 100 \end{bmatrix}} \cdot \underset{\boldsymbol{\psi}}{\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}}$$

We can solve this system for ψ_1 and ψ_2 to obtain ψ_1 = .3667 and ψ_2 = .5333. The call value is:

$$c_0 = [.3667 \quad .5333]\begin{bmatrix}0\\20\end{bmatrix} = 10.667$$

The risk-neutral probabilities (synthetic probabilities) are determined as follows:

$$q_1 = \frac{.3667}{.3667 + .5333} = .4074 \qquad q_2 = \frac{.5333}{.3667 + .5333} = .5926$$

Observe that with respect to the risk-neutral probabilities, the expected value of stock K is: $E[K]=q_1K_1+q_2K_2$ = .4074×20 +.5926×80 = 55.556, which gives an expected return of (55.556-50)/50 = 11.11%. The return on the riskless T-bill is: (100-90)/90 = 11.11%. The returns are equal, thus illustrating the risk-neutral nature of the arbitrage free pricing mechanism.

We can also use the concept of arbitrage to directly value the call. Since the call and bond are priced, and their payoff vectors of the stock and T-bill span the 2-outcome space, they form complete capital markets. Thus, a portfolio comprising the stock and T-bill can replicate the payoff structure of the call:

$$\begin{bmatrix}20 & 100\\80 & 100\end{bmatrix}\begin{bmatrix}\gamma_q\\ \gamma_b\end{bmatrix} = \begin{bmatrix}0\\20\end{bmatrix}$$

$$\begin{bmatrix}-.016667 & .016667\\ .013333 & -.003333\end{bmatrix}\begin{bmatrix}0\\20\end{bmatrix} = \begin{bmatrix}.3333\\-.06667\end{bmatrix}$$

Thus, the payoff structure of the call is replicated with .3333 shares of underlying stock and -.06667 riskless bonds. This portfolio requires a net investment of .3333×50+(-.06667)×90= \$10.67, so the call must be worth \$10.67.

Put-Call Parity: One Time Period

A *put* is an option that grants its owner the right to sell the underlying stock at a specified exercise price on or before its expiration date. Consider a European put (European options can be exercised only at expiration) with value p_0 and a European call with value c_0 written on the same underlying stock currently priced at S_0. Both options have exercise prices equal to X and expire at time T. The riskless return rate is r. Since the payoff function of the call at expiration is c_T = MAX[S_T-X, 0] and the payoff function for the put is p_T = MAX[X-S_T, 0], the following n-outcome system describes the pricing of a put:

$$(31) \qquad \begin{bmatrix}P_1\\P_2\\\vdots\\P_n\end{bmatrix} = -\begin{bmatrix}S_1\\S_2\\\vdots\\S_n\end{bmatrix} + \begin{bmatrix}X\\X\\\vdots\\X\end{bmatrix} + \begin{bmatrix}C_1\\C_2\\\vdots\\C_n\end{bmatrix}$$

$$\mathbf{p} = -\,\mathbf{s} + \mathbf{x} + \mathbf{c}$$

$$\text{Max}[X\text{-}S,0] = -\ S + X + \text{Max}[S\text{-}X,0]$$

This *put–call parity* relation holds regardless of the number of potential outcomes in the state space. The payoff structures of the European call, underlying stock and bond will always be sufficient to replicate the European put as long as both the put and call expire when the riskless debt matures and the put and call have the same striking price as the face value of the debt. Consider the following numerical example where there are three potential stock prices, 120, 100 and 80 and a 105 exercise price for the options:

$$\begin{bmatrix} 0 \\ 5 \\ 25 \end{bmatrix} = -\begin{bmatrix} 120 \\ 100 \\ 80 \end{bmatrix} + \begin{bmatrix} 105 \\ 105 \\ 105 \end{bmatrix} + \begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{p} \quad = \quad \mathbf{-S} \quad + \quad \mathbf{x} \quad + \quad \mathbf{c}$$

Because the put–call parity relation must hold at option expiry regardless of the underlying stock price, the following put–call parity relation must hold at option expiry date time T:

$$p_T = -S_T + X + c_T$$

Similarly, one of the following put-call parity relations must hold at time zero, depending on exactly how bonds are valued relative to the riskless rate (with discrete or continuous compounding):

(32) $$p_0 = -S_0 + X(1+r)^{-T} + c_0 \qquad\qquad p_0 = -S_0 + Xe^{-rT} + c_0$$

<u>Completing the State Space</u>

When a set of priced payoff vectors forms the basis for the n-dimensional state space, any security in that economy can be replicated and priced as a linear combination of those payoff vectors and prices. This means that any security in an n-state economy can be priced if n other securities reflected by a linearly independent set of payoff vectors are priced. Derivative securities have payoff vectors that are contingent on the payoff vectors for other securities, one can define the outcome space relative to the payoff vector for the underlying security. One can create unlimited numbers of different derivative securities such as options on stocks or other existing assets. This is possible because, for example, unlimited numbers of options, all with different exercise prices can be created and marketed on every underlying security. When this potentially unlimited number of options are marketed and priced, the state space will surely be spanned. That is, with an assured sufficiently large numbered set of linearly independent payoff vectors, the state space can be spanned with the underlying security, a riskless bond and all the options written on that security. Thus, the bond, the underlying security and options written on that security can form the basis for the n-potential outcome economy. In fact, the bond can be replaced with another option. Consider a stock that will pay either 20, 40 or 60. Two calls are written on that stock, one with an exercise price of 30 and a second with an exercise price of 50:

$$\begin{bmatrix} 20 \\ 40 \\ 60 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 10 \\ 30 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$\mathbf{S} \qquad \mathbf{c_{30}} \qquad \mathbf{c_{50}}$$

These three securities form the basis for the 3-dimensional state space. Thus, their cash flows can be used to replicate any security's cash flow in this 3-outcome payoff space. For example, consider a call option with an exercise price of 40. This option with a payoff vector of $[0, 0, 20]^T$ can be replicated with a portfolio with payoff vectors forming the basis as follows:

$$\begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 0 \\ 60 & 10 & 0 \\ 0 & 30 & 10 \end{bmatrix} \begin{bmatrix} s \\ c_{30} \\ c_{50} \end{bmatrix}$$

$$\mathbf{c_{40}} \;=\; \mathbf{CF}^{T} \quad \lambda$$

Since the cash flow structure of this call can be replicated with the cash flow structures of the three securities in the 3-outcome basis, it can be priced as a linear combination of the prices of the three securities forming the basis. Solving for $\boldsymbol{\gamma}$ where $\boldsymbol{\gamma}=(\mathbf{CF}^T)^{-1}\times \boldsymbol{c_{40}}$, leads to $\boldsymbol{\gamma}^T=[0,0.2]$. This means that the call is replicated with 0 shares of stock, 0 calls with exercise price equal to 30 and 2 calls with exercise price equal to 50. Thus, the price of the X=40 call should be twice that of the X=50 call.

More generally, one should always be able to form a basis for an *n*-state economy with a stock, a riskless bond and *n-2* priced options written on that stock. Any other security (usually other options on that stock) whose payoff vectors are in the same payoff space can be priced as a linear combination of the prices of the securities forming the basis of payoff vectors for that economy. Thus, we should always be able to create complete capital markets by trading and pricing the appropriate number of calls in that economy.

E. Discrete Time-State Models

Discrete Time Models

Discrete time models allow for state variables to change only at a countable number of points in time as opposed to continuous time models that allow for state variables to change in a continuous manner over an infinite number of intervals in any time span. Many people find discrete models more intuitively appealing, but calculus enables continuous models to be easier to implement in many circumstances.

Discrete Time and State Models

We have introduced single outcome models in multiple period environments and we have introduced uncertainty (multiple states) in single-period environments. Now, we will introduce multiple states into multiple-period environments. However, the methodologies and even subscripting that we used become very cumbersome very quickly. Nevertheless, we do need to be able to understand and work in multiple period frameworks with uncertainty. We offer the material in this section not to suggest that the reader actually use this type of methodology to price securities in uncertain multiple period environments, but just to provide some intuition on how securities evolve in such environments. We will introduce methodologies in Chapter 6 that will be far more useful for pricing securities in these more complex environments.

The full set of pure securities can be synthesized and priced for any time period as long as complete capital markets exist for that period. However, complete capital markets will require that there exist at least as many securities producing a linearly independent set of payoff vectors as there exist outcomes.

Multiple Time Periods and States: Illustration

Suppose that there exist three securities x, y and b_2 (the 2-year riskless bond) in a 2-period economy with the following payoff vectors for period 2:

$$\mathbf{x} = \begin{bmatrix} 4 \\ 18 \\ 25 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 5 \\ 7 \\ 10 \end{bmatrix} \quad \mathbf{b_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

with prices at time 0: $S_{x,0} = 11$, $S_{y,0} = 5.696$ and $B_{0,2} = .907$ for security x, y and the riskless bond, respectively. The value $B_{0,2}$ refers to the time 0 value of the bond that is worth 1 at time 2.These given prices are depicted in the upper panel of Figure 1. In Figure 1, the symbols u and d refer to a security price upjump or downjump. Similarly, the notation $\{u,u\}$ refers to consecutive upjumps over two periods while $\{u,d\}$ refers to a price upjump followed by a downjump from times 0 to 2. The notations $\{d,u\}$ and $\{d,d\}$ are defined analogously. The three possible states in this framework ending at time 2 are $\omega_{u,u}$, $\omega_{u\wedge d}$ and $\omega_{d,d}$. Note that there are 4 outcomes in this recombining binomial framework: $\{u,u\}$, $\{u,d\}$, $\{d,u\}$and $\{d,d\}$, where $\{\{u,d\}, \{d,u\}\}$ combine and form a single state $\omega_{u\wedge d}$. With three potential states and three securities that span this state space in this binomial framework, we can price the three 2-period pure securities at time zero. Time zero values for each of the 3 pure securities that pay off in period 2 are obtained from the following:

$$\begin{bmatrix} 11 \\ 5.696 \\ .907 \end{bmatrix} = \begin{bmatrix} 4 & 18 & 25 \\ 5 & 7 & 10 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \psi_{0,2;d,d} \\ \psi_{0,2;u\wedge d} \\ \psi_{0,2;u,u} \end{bmatrix}.$$

Thus, we find that our time zero 2-period pure security prices are $\psi_{0,2;d,d}$=.407, $\psi_{0,2;\,u\wedge d}$ =.446, $\psi_{0,2;u,u}$=.054.[9] These computed prices are depicted in the lower panel of Figure 1.

Suppose that none of these securities make a payment at time one. However, for purposes of our illustration, we still wish to observe the price evolution of our securities over time. Suppose that the riskless rate will be .05 in the first period, which implies that the riskless bond worth 1 at time 2 has a time one value of 1/1.05=.952. Further suppose that at time 1, security x will be worth either $S_{x,1;d} = 6$ (security x price, time 1 outcome ω_d) or $S_{x,1;u} = 18$, and that these time 1 prices will lead to the time 2 prices of either 4, 18 or 25 in the manner depicted in the upper panel of Figure 1. Now for time period 1, we have two priced securities associated with each state, completing our markets in each state for that period, and allowing us to price pure securities and security y in the lower panel of Figure 1. In a complete market, arbitrage free pricing guarantees that any independent set of prices that we pick to accomplish the pricing will result in consistent pricing. Even though we have already priced the securities going from time 0 to 2, if we now price securities going from time 1 to 2 and them from time 0 to 1, the pricing will be consistent. Our pure securities are priced at time 1 by solving the two systems, one for each of

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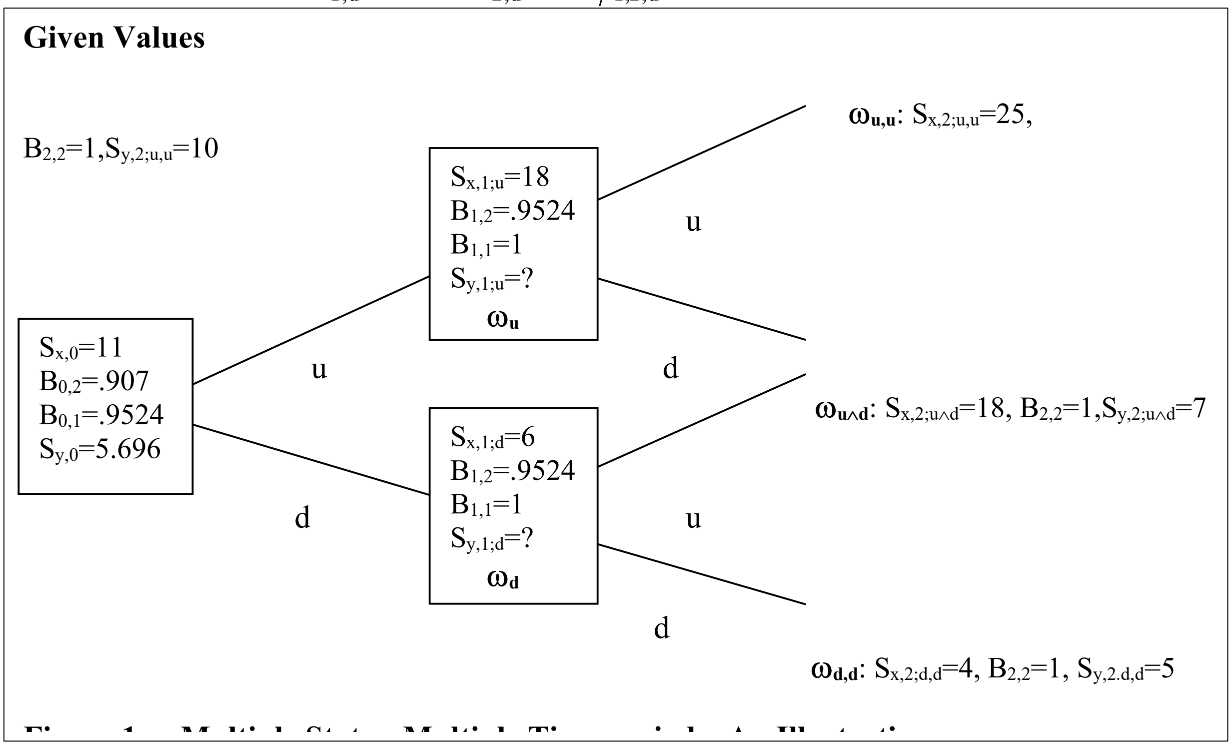
[9] $\psi_{0,2;u,u}$ reads: the time 0 price of pure security for state $\omega_{u,u}$, paying off at time 2. In general when pricing pure securities, the first subscript refers to the time the security is being priced, the second subscript is when the pure security is paid off, and after the semicolon is the state or outcome(s). $S_{x,2;u,u}$ is the price of security x at time 2 if it is in state $\omega_{u,u}$.

two possible states at time 1:[10]

$$\begin{bmatrix} 6 \\ .9524 \end{bmatrix} = \begin{bmatrix} 4 & 18 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \psi_{1,2;d,d} \\ \psi_{1,2;d,u} \end{bmatrix} \quad \text{State } \omega_d \text{ Time 1 Prices}$$

$$\mathbf{v_{1;d}} = \mathbf{CF_{2;d}} \cdot \boldsymbol{\psi_{1,2;d}}$$



[10] $\psi_{1,2;d,u}$ reads: the time 1 price of pure security for state $\omega_{u\wedge d}$, paying off at time 2, contingent on realizing state ω_d at time 1. Note: Remember that u∧d includes both {u,d} and {d,u}.

Computed Values

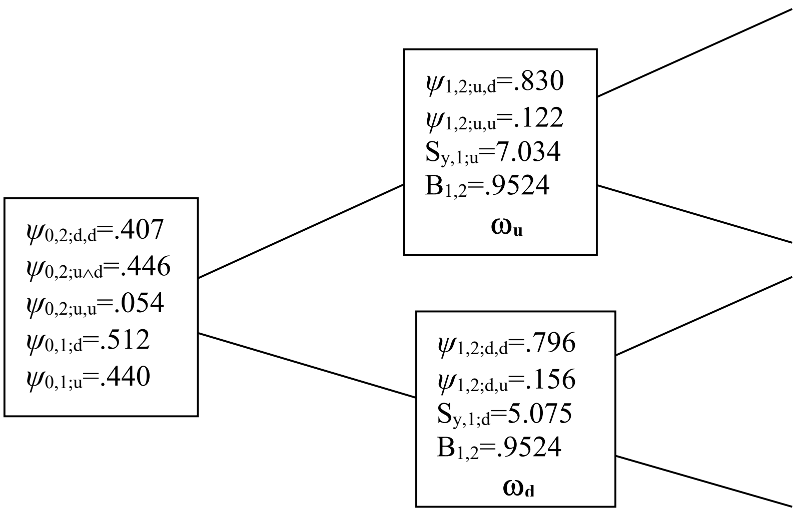


Figure 1.b: Multiple States, Multiple Time periods: An Illustration

$$\underset{\mathbf{v_{1;u}}}{\begin{bmatrix} 18 \\ .9524 \end{bmatrix}} = \underset{\mathbf{CF_{2;u}}}{\begin{bmatrix} 18 & 25 \\ 1 & 1 \end{bmatrix}} \cdot \underset{\boldsymbol{\psi_{1,2;u}}}{\begin{bmatrix} \psi_{1,2;u,d} \\ \psi_{1,2;u,u} \end{bmatrix}} \quad \text{State } \omega_u \text{ Time 1 Prices}$$

We solve these two systems and find that: $\psi_{1,2;d,d}$=.796, $\psi_{1,2;d,u}$=.156, $\psi_{1,2;u,d}$=.830 and $\psi_{1,2;u,u}$=.122. Thus, contingent on whether state 1 or state 2 is realized, we will know pure security prices for instruments that pay off in period 2. We can also use this information above to price security y at time 1 to be either $.796\times5 + .156\times7 = 5.075 = S_{Y,1;d}$ or $.830\times7 + .122\times 10 = 7.034 = S_{Y,1;u}$ depending on whether security x decreased (d) or increased (u) in value from time 0 and 1.

Even though none of our priced securities have time one payoffs, we can still use our time one prices and value pure securities that pay off at time 1. Since only two states are possible at time 1, we can compute two pure security prices based on the time one *2*-year bond value of .9524 and the security x value of either 6 or 18:[11]

$$\underset{\mathbf{v_0}}{\begin{bmatrix} 11 \\ .907 \end{bmatrix}} = \underset{\mathbf{CF_1}}{\begin{bmatrix} 6 & 18 \\ .9524 & .9524 \end{bmatrix}} \cdot \underset{\boldsymbol{\psi_{0,1}}}{\begin{bmatrix} \psi_{0,1;d} \\ \psi_{0,1;u} \end{bmatrix}}$$

Thus, Time 0 pure security prices are $\psi_{0,1;d}$=.512 and $\psi_{0,1;u}$=.441. These enable us to price at time zero any security that pays or has a known value in time 1.

This section serves as a very brief introduction to multiple state-multiple time discrete pricing. In Chapters 7 and 8, we will study the pricing mechanism more systematically along

[11] $\psi_{0,1;d}$ reads: the pure security time 0 price, when payment at time 1 is made by the pure security for state ω_d.

with its connection to martingales. We will generalize these ideas in Chapter 3, 7 and 9 to continuous processes. In particular, we will deal with these problems in Chapters 7, 8 and 9 by assuming that our uncertainty can be represented with specific stochastic processes (e.g., binomial or Brownian motion frameworks).

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Exercises

1. What is the present value of a security to be discounted at a ten percent rate promising to pay $10,000 in:

   a. twenty years?
   b. ten years?
   c. one year?
   d. six months?
   e. seventy three days?

2. The Foxx Company is selling preferred stock which is expected to pay a fifty dollar annual dividend per share. What is the present value of dividends associated with each share of stock if the appropriate discount rate were eight percent and its life expectancy were infinite?

3. What would be the present value of $10,000 to be received in twenty years if the appropriate discount rate of 10% were compounded:

   a. annually?
   b. monthly?
   c. daily?
   d. continuously?

4.a. What would be the present value of a 30-year annuity if the $1000 periodic cash flow were paid monthly? Assume a discount rate of 10% per year.
b. Should an investor be willing to pay $100,000 for this annuity?
c. What would be the highest applicable discount rate for an investor to be willing to pay $100,000 for this annuity?

5.a. Suppose that a $1000 face value bond will make a single interest payment at an annual rate of 5%. Suppose this bond is currently selling for 102 (actually meaning 102% of its face value, or 1020) and that it matures in one year when its coupon payment is made. What is the one-year spot rate?
b. Now, drawing on your results from part a of this problem, consider a second $1000 face value two-year bond making interest payments at an annual rate of 5%. Suppose this bond is currently selling for 101.75 (meaning 101.75% or 1017.5) and that it matures in two years when its second coupon payment is made. What is the two-year spot rate implied by this bond, considering the one-year spot rate?
c. What is the three-year spot rate $y_{0,3}$ implied by the three-year 5% coupon bond priced at 101.5 based on parts a and b of this question?

In parts d through g, assume that the pure expectations theory applies.
d. What is the one-year forward rate on a loan originating in one year?
e. What is the one year forward rate on a loan originating in two years?
f. What is the two-year forward rate on a loan originating in one year?
g. Map out the yield curve based on your answers from the preceding parts of this problem.

6. Suppose we toss a die and observe the outcome. We can obtain an outcome of 1, 2, 3, 4, 5, or 6 of the die. If each outcome is equally likely, we would assign a probability of 1/6 to each of the

outcomes.
a. What is the sample space for numerical values of the roll of the die?
b. How many events are in Φ?
c. What is P({2,4,6})

7. Suppose we toss three fair coins and observe the result. By a fair coin, we mean that there is a 50-50 chance that the coin comes out either heads or tails.
a. List the sample space (all possible outcomes) for this experiment.
b. How many possible events are in the event space for this experiment?
c. Is {HHH,TTT} an event in this event space? If so, what is its probability?
d. Is ∅ an event in this event space? If so, what is its probability?

8. In U.S. stock markets, insiders are permitted to trade (provided they register their trades), but only when their trades are not motivated by inside information. To evade detection, insiders often engage in activity that lead to particular trading patterns that the New York Stock Exchange (NYSE) uses software systems to detect. However, before implementing a new software system, the NYSE must ensure that the system does not signal an excessive rate of false positives; that is, the new system should not signal too many incidences in which traders are falsely accused of illegal insider trading.

Suppose that 1% of all NYSE trades are known to be motivated by the illegal use of inside information.[12] Further suppose that the NYSE is considering implementation of a new software system for detecting illegal insider trading. In this proposed system, a positive signal from the system has a 90% probability of leading to a conviction for illegal insider trading. That is, 90% of all signals of insider trading are considered to be truthful signals and the other 10% of positive signals are taken to be false based on no conviction. Thus, we assume that a conviction for insider trading only occurs when an illegal act has been committed as alleged. Further, assume that among all trades, that there is a 5% probability that the system signals positive for an illegal trade; obviously, some proportion of these signals must be false positives. Based on this information, what is the probability that a given positive signal by the proposed system for insider trading actually results in a conviction? Based on your calculation, what is the probability that the system's signal for illegal trading is false?

9. Suppose that securities x and y exist in a 2-potential outcome world and have the following payoff vectors:

$$\mathbf{x} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$$

Suppose that security x has a market value of 5 and security y has a market value of 8.
a. Is the set of 2 payoff vectors linearly independent?
b. Do these two payoff vectors span the 2-dimensional space?
c. Is there a complete capital market in this economy?
d. What are the prices of pure securities in this economy?
e. What is the price of a third security z with the following payoff structure:

$$\mathbf{z} = \begin{bmatrix} 20 \\ 8 \end{bmatrix}$$

f. Calculate risk-neutral probabilities for each of the two states.

[12] All numbers and details provided in this illustration are fictitious.

10. Security A will pay 5 in outcome 1, 7 in outcome 2 and 9 in outcome 3. Security B will pay 2 in outcome 1, 4 in outcome 2 and 8 in outcome 3. Security C will pay 9 in outcome 1, 1 in outcome 2 and 3 in outcome 3. Both securities A and C currently sell for $5 and Security B currently sells for $3. What would be the value of Security D which will pay 1 in each of the 3 outcomes?

11. Rollins Company stock currently sells for $12 per share and is expected to be worth either $10 or $16 in one year. The current riskless return rate is .125. What would be the value of a one-year call with an exercise price of $8?

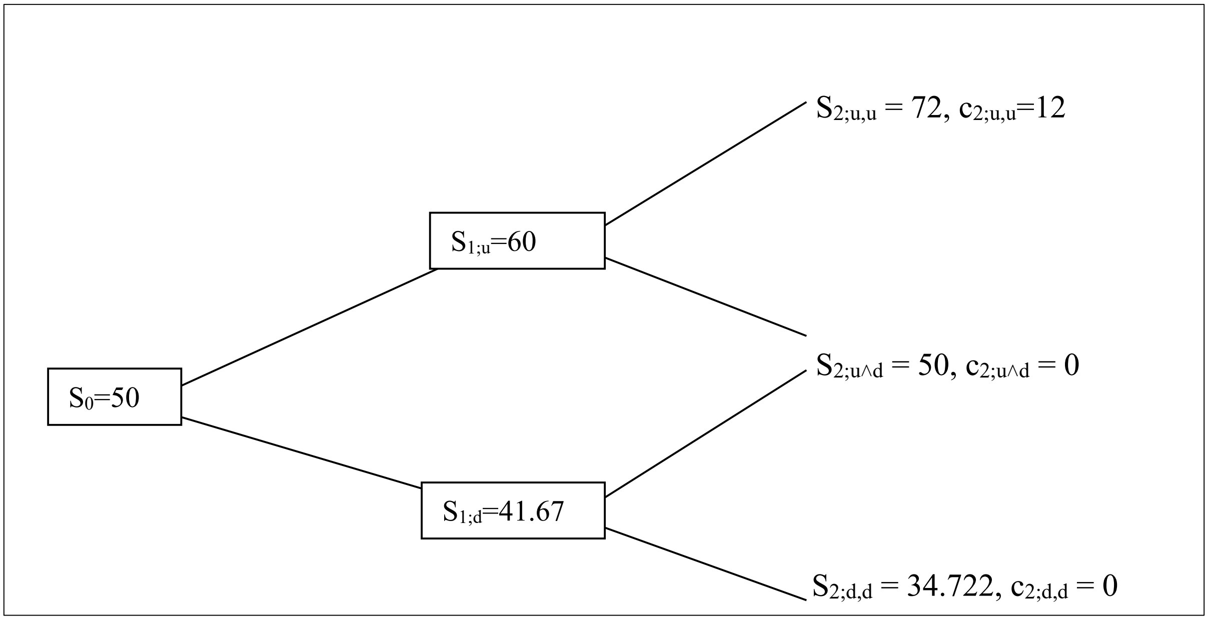
12. In a one time-period binomial economy with a riskless return rate equal to 10%, a stock currently sells for $50. Its potential terminal prices are either $80 or $40. What are the synthetic (risk-neutral) probabilities for this economy?

13. Harper Company stock currently sells for $14 per share and is expected to be worth either $10, $16 or $25 in one year. The current riskless return rate is .125. A one-year call with an exercise price of $15 currently sells for $3.
a. What would be the value of a one-year call with an exercise price of $9?
b. What are the synthetic probabilities in this economy?

14. Buford Company stock currently sells for $24 per share and is expected to be worth either $20 or $32 in one year. The current riskless return rate is .125. What would be the value of a one-year call with an exercise price of $16?

15. Robinson Company stock currently sells for $20 per share and will pay off either $15 or $25 in one year. A one-year call with an exercise price equal to $18 has been written on this stock. This call sells for $7.
a. What is the value of a one-year call with an exercise price equal to $22?
b. What is the riskless return rate for this economy?
c. What is the value of a one-year put that can be exercised for $40?

16. Consider a two 6-month time period framework depicted in the following figure in which Company *J* stock currently sells for $50 per share and a riskless $100 face value T-bill currently sells for $90 ($d_2 = .9$). Assume that the discount rate is the same for each 6-month period. There are 2 possible outcomes for each 6-month period contingent on the outcomes that preceded them. At the end of the year (That is, two 6-month periods), Company *J* stock will pay its owner either $34.722, $50 or $72 per share. Note that these payments are $(1/1.2^{2})\times 50$, $(1.2/1.2)\times 50$ and $1.2^2\times 50$; that is, the stock price will decrease either by 30.56%, stay the same or increase by 44% over the course of the year. We will assume that proportional increases and decreases in the stock's price are the same in each six-month interval during the year; that is, each is by the factor $\sqrt{1.44}=1.2$. The stock will make no payments to shareholders (no dividends) at the end of 6 months. A one-year call with an exercise price of $60 trades on *J* stock shares. This call will be worth either $0 or $12 when it expires, based on the value of the underlying stock.



a. Write one-year payoff vectors for the stock, T-bill and the call.

b. Assume that risk neutral probabilities associated with upjumps are constant over the two 6-month intervals during the year in this framework. Are markets complete in this 4-outcome 2-period market? Why or why not?

c. Assume that risk neutral probabilities associated with downjumps are constant over the two 6-month intervals during the year as are riskless rates in this framework. What is the pure security price associated with each upjump? What is the pure security price associated with each downjump?

d. From your answers to a, b and c above, find risk neutral probabilities for the first 6-month interval, again, assuming that the 6-month riskless interest rate is $r = .0541$. That is, find $q_{0,1;d}$ and $q_{0,1;u}$.

e. What are the three 2-period pure security prices $\psi_{0,2;d,d}$, $\psi_{0,2;u\wedge d}$ and $\psi_{0,2;u,u}$?

f. What is the current value of the call?

Solutions

1. a. PV = $10{,}000/1.1^{20}$ = 10,000/6.7275 = 1486.44
   b. PV = $10{,}000/1.1^{10}$ = 10,000/2.5937425 = 3855.43
   c. PV = $10{,}000/1.1^{1}$ = 10,000/1.1 = 9090.91
   d. PV = $10{,}000/1.1^{.5}$ = $10{,}000/1.1^{.5}$ = 10,000/1.0488088 = 9534.63
   e. PV = $10{,}000/1.1^{.2}$ = 10,000/1.0192449 = 9811.18; Note: 73 days is .2 of one year

2. PV_p = CF/y = 50/.08 = 625

3. a. PV = $10{,}000/1.1^{20}$ = 10,000/6.7275 = 1486.44
   b. PV = $10{,}000/(1+.1/12)^{12\times 20}$ = 10,000/7.328074 = 1364.62
   c. PV = $10{,}000/(1+.1/365)^{365\times 20}$ = 10,000/7.3870322 = 1353.72
   d. PV = $10{,}000 \times e^{-.1\times 20}$ = 1353.35

4. a. First, the monthly discount rate is .1÷12 = .008333333

$$PV = \frac{1{,}000}{.00833333}\left[1 - \frac{1}{(1+.00833333)^{360}}\right] = \$113{,}951$$

   b. Yes, since the PV = $113,951 exceeds the $100,000 price
   c. $100{,}000 = \frac{1{,}000}{r/12}\left[1 - \frac{1}{(1+r/12)^{360}}\right]$

Solve for r; by process of substitution, we find that r = .11627. One can also find r by using Newton's Method for approximating roots, covered in any standard calculus text. We also cover this method in section 7.4.2, or one can use the method of bisection covered in section 7.4.1.

5.a. Let d_t be the discount function for time t; that is, $d_t = 1/(1+r_{0,t})^t$:

$$PV = \sum_{t=1}^{n} cF \cdot d_t + F \cdot d_n$$

where *c* is the bond coupon rate. When d_t is calculated, we obtain the spot rate as follows:

$$r_{0,t} = (1/d_t)^{1/t} - 1$$

In part a, we have the following values *n=1, cF*=.05×1,000=50, *F*=1000, and *PV*=1020. Plugging this information into the PV equation above, we get:

$$1020 = 50d_1 + 1000d_1 = 1050d_1.$$

$$d_1 = 1020/1050 = .97143;\ r_{0,1} = (1/d_1)^{1/1} - 1 = (1/.97143) - 1 = .0294$$

Thus, the one-year spot rate is 2.94%.

b. In part b, we have the following values *n*=2, *cF*=50, *F*=1000, d_1=.97143, and *PV*=1017.5. Using the PV equation with *n=2,* the two-year spot rate implied by this bond is bootstrapped from the one-year bond as follows:

$$1017.5 = 50 \cdot .97143 + 50 \cdot d_2 + 1000 \cdot d_2$$

$$1050d_2 = 968.93$$

$$d_2 = .92279$$

$$r_{0,2} = (1/.92279)^{1/2} - 1 = .0410$$

Bootstrapping is making use of the rate (the one-year rate) or information that is already known to obtain the desired result (the two-year rate).

c. Similarly, the three-year spot rate $r_{0,3}$ is bootstrapped from the one-year and two-year bonds as follows:

$$1015.0 = 50 \cdot .97143 + 50 \cdot .92279 + 50 \cdot d_3 + 1000 \cdot d_3$$
$$1050d_3 = 920.29$$
$$d_3 = .87647$$
$$r_{0,3} = (1/.87647)^{1/3} - 1 = .0449$$

d. If we accept the Pure Expectations Theory for the term structure of interest rates, we can obtain forward rates from spot rates. We can use equation (12) with $n=2$ to find the one-year forward rate $r_{1,2}$ originating in one year. The point is equation (12) allows us to find this forward rate since the two-year spot rate is a function of the one-year spot rate and the one-year forward rate on a loan:

$$r_{0,2} = .0410 = \sqrt{(1+r_{0,1})(1+r_{1,2})} - 1 = \sqrt{(1+.0294)(1+r_{1,2})} - 1$$

We can use this relationship to solve for the one-year forward rate on a loan originating in one year as follows:

$$(1+.0410)^2 = (1+.0294)(1+r_{1,2})$$

$$r_{1,2} = \frac{(1+.0410)^2}{(1+.0294)} - 1 = .0527$$

e. Similarly, we can solve for the one year forward rate on a loan originating in two years, forward rate $r_{2,3}$ as follows:

$$(1+.0449)^3 = (1+.0294)(1+.0527)(1+r_{2.3})$$

$$r_{2,3} = \frac{(1+.0449)^3}{(1+.0294)(1+.0527)} - 1 = .0528$$

f. The two-year forward rate on a loan originating in one year, forward rate $r_{1,3}$ is determined as follows. We know that

$$(1+r_{0,3})^3 = (1+r_{0,1})(1+r_{1,2})(1+r_{2,3})$$

and

$$(1+r_{1,3})^2 = (1+r_{1,2})(1+r_{2,3}).$$

Substituting the left side of the equation immediately above into the right side of the equation above that, we obtain:

$$(1+r_{0,3})^3 = (1+r_{0,1})(1+r_{1,3})^2.$$

We have already calculated the values: $r_{0,3} = .0449$ and $r_{0,1} = .0294$. Thus, we have:

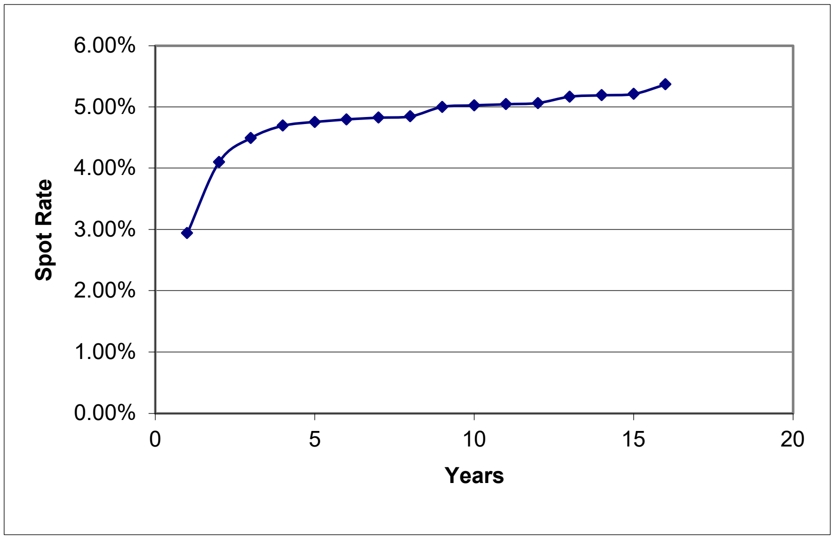
$$(1+.0449)^3 = (1+.0294)(1+r_{1,3})^2$$

$$r_{1,3} = \sqrt{\frac{(1+.0449)^3}{(1+.0294)}} - 1 = .0527$$

which is identical to the following alternative calculation:

$$r_{1,3} = \sqrt{(1+r_{1,2})(1+r_{2,3})} - 1$$
$$r_{1,3} = \sqrt{(1+.0527)(1+.0528)} - 1 = .0527$$

g. The yield curve is the graph of $r_{0,T}$ versus T for $T = 1,2,3,\ldots$ From parts a, b, and c, the following is the yield curve:



Note that our graph includes extra points beyond our results from parts a, b, and c, extending out 16 years based on data not presented here.

6.a. The sample space is Ω = {1,2,3,4,5,6}.

b. There are 2^6=64 possible subsets of Ω.

c. By rule 3 of the requirement to be a probability, we have P({2,4,6}) = P({2}) + P({4}) + P({6}) = 1/6+1/6+1/6 = 1/2. Thus, the probability of obtaining an even number is ½.

7.a. The individual outcomes are all of the possible 3 coin outcomes. The sample space Ω={HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}, where H means heads and T means tails.

b. There are 8 outcomes, meaning that there are 2^8 = 256 possible events including all possible combinations of outcomes and ∅.

c. Yes; 1/8+1/8 = 1/4

d. Yes; 0

8. This problem seems complicated at first glance. However, Bayes Rule will simplify its calculation. Let *A* be the event that a trade tests positive for being initiated illegally and let *B* be the event that a trade actually is illegally motivated by inside information. Thus, we have the given information that P[A|B]=.9, P[B]=.01, and P[A]=.05. We seek to find:

$$P[B|A] = \frac{P[A|B]P[B]}{P[A]} = \frac{.9(.01)}{.05} = .18.$$

Thus, there is an 18% probability that a trade indicated by the system to be illegal actually was illegal. That is, 82% of signals of illegal trading are actually false. This result may seem surprising at first, because among illegal trades, the test correctly gives a positive signal 90% of the time. Yet only 18% of the trades that test positive on this new system for being motivated by inside information are actually illegally motivated. The intuition is as follows. Even though the signal is positive only 5% of the time among all trades, illegal trades comprise only 1% of the population of all trades. Thus, most of the trades that signal positive for being illegally motivated cannot lead to a conviction.

9.a. These two payoff vectors are linearly independent. They can be stacked into a 2×2 matrix, and this matrix can be inverted.

b. Yes, because the set of 2 vectors is linearly independent in 2-dimensional space.

c. Yes. Vectors **x** and **y** span the two-dimensional vector space and x and y are priced. Thus, there exists a complete capital market in this economy. This means that every other security in this economy is a linear combination of the payoff vectors for securities x and y and can be priced.

d. In the absence of arbitrage opportunities, a portfolio comprising 8 of pure security 1 and 4 of pure security 2 must also have a market value of 5 and a portfolio comprising 2 of pure security 1 and 10 of pure security 2 must have a market value of 8:

$$\begin{bmatrix}5\\8\end{bmatrix} = \begin{bmatrix}8 & 4\\2 & 10\end{bmatrix}\cdot\begin{bmatrix}\psi_1\\\psi_2\end{bmatrix}$$

$$\mathbf{v} \quad = \quad \mathbf{CF} \quad \boldsymbol{\psi}$$

We solve this system for ψ_1 and ψ_2 to obtain $\psi_1 = .25$ and $\psi_2 = .75$.

e. The price of market security *z* is determined from the prices of our two pure securities:

$$S_{z,0} = [.25 \quad .75]\begin{bmatrix}20\\8\end{bmatrix} = 11$$

f. .25/1 = .25 and .75/1 = .75

10. First, find the values of Pure Securities 1, 2 and 3 as follows:

$$\begin{bmatrix}5 & 7 & 9\\2 & 4 & 8\\9 & 1 & 3\end{bmatrix}\begin{bmatrix}\psi_1\\\psi_2\\\psi_3\end{bmatrix} = \begin{bmatrix}5\\3\\5\end{bmatrix}$$

$$\begin{bmatrix}.0227272 & -.0681818 & .11363636\\.375 & -.375 & -.125\\-.1931818 & .3295454 & .03409090\end{bmatrix}\begin{bmatrix}5\\3\\5\end{bmatrix} = \begin{bmatrix}\psi_1\\\psi_2\\\psi_3\end{bmatrix} = \begin{bmatrix}.47727\\.125\\.19318\end{bmatrix}$$

$$\boldsymbol{CF^{-1}} \qquad \mathbf{v} = \boldsymbol{\psi} = \boldsymbol{\psi}$$

Thus, we find that $\psi_1 = .47727$, $\psi_2 = .125$ and $\psi_3 = .19318$. The value of Security D equals 1 × .47727+ 1 × .125 + 1 × .19318 = .79545.

11. Since the riskless return rate is .125, the current value of a security guaranteed to pay $1 in one year would be $1/1.125 = .88889. The security payoff vectors are as follows:

$$S = \begin{bmatrix}10\\16\end{bmatrix} \quad b = \begin{bmatrix}1\\1\end{bmatrix} \quad c = \begin{bmatrix}2\\8\end{bmatrix}$$

Portfolio holdings are determined as follows:

$$\begin{bmatrix}10 & 1\\16 & 1\end{bmatrix}\cdot\begin{bmatrix}\gamma_S\\\gamma_b\end{bmatrix} = \begin{bmatrix}2\\8\end{bmatrix}$$

The following includes the inverse matrix:

$$\begin{bmatrix}\gamma_S\\\gamma_b\end{bmatrix} = \begin{bmatrix}-.166667 & .166667\\2.66667 & -1.66667\end{bmatrix}\cdot\begin{bmatrix}2\\8\end{bmatrix} = \begin{bmatrix}1\\-8\end{bmatrix}$$

We find that $\gamma_S = 1$ and $\gamma_b = -8$. This implies that the payoff structure of a single call can be replicated with a portfolio comprising 1 share of stock and short-selling 8 T-Bills. This portfolio requires a net investment of $1\times12-8\times.88889 =$ \$4.89. Since the call has the same payoff structure as this portfolio, its current value must be \$4.89.

13. From problem 10, we see that u = 80/50 = 1.6, d = 40/50 = .8, and q = (1+r - d)/(u-d) = (1+.1 - .8)/(1.6 - .8) = .375

13.a. Since the riskless return rate is .125, the current value of a security guaranteed to pay \$1 in one year would be \$1/1.125 = .88889. The security payoff vectors are as follows:

$$h = \begin{bmatrix}10\\16\\25\end{bmatrix}, b = \begin{bmatrix}1\\1\\1\end{bmatrix}, c_{15} = \begin{bmatrix}0\\1\\10\end{bmatrix}, c_9 = \begin{bmatrix}1\\7\\16\end{bmatrix}$$

Portfolio holdings are determined as follows:

$$\begin{bmatrix}10 & 1 & 0\\16 & 1 & 1\\25 & 1 & 10\end{bmatrix}\cdot\begin{bmatrix}\gamma_h\\\gamma_b\\\gamma_{c15}\end{bmatrix}=\begin{bmatrix}1\\7\\16\end{bmatrix}$$

The following includes the inverse of the securities payoff matrix:

$$\begin{bmatrix}\gamma_h\\\gamma_b\\\gamma_{c15}\end{bmatrix}=\begin{bmatrix}-.2 & .22222 & -.02222\\3 & -2.2222 & .22222\\.2 & -.33333 & .133333\end{bmatrix}\cdot\begin{bmatrix}1\\7\\16\end{bmatrix}=\begin{bmatrix}1\\-9\\0\end{bmatrix}$$

We find that a portfolio replicating a call with an exercise price of 9 can be constructed with the following numbers of shares, T-Bills and calls with an exercise price of 15: $\gamma_h = 1$ and $\gamma_b = -9$ and $\gamma_{c15} = 0$. Thus, the payoff structure of a single call with an exercise price of 9 can be replicated with a portfolio comprising 1 share of stock, short-selling 9 T-Bills, and selling 0 calls with an exercise price of 15. This portfolio requires a net investment of 1×14-9×.88889 + 0×15 = \$6. Since the call has the same payoff structure as this portfolio, its current value must be \$6.

b. To find the pure security prices, we need to solve the matrix equation:

$$\begin{bmatrix}10 & 16 & 25\\1 & 1 & 1\\0 & 1 & 10\end{bmatrix}\begin{bmatrix}\psi_1\\\psi_2\\\psi_3\end{bmatrix}=\begin{bmatrix}14\\.88889\\3\end{bmatrix}.$$

Pure security prices are $\psi_1 = .46667$, $\psi_2 = .1358$ and $\psi_3 = .28642$. Thus, synthetic probabilities are .46667/.88889 = .525, .1358/.88889 =.1528 and .28642/.88889 = .3222.

14. Since the riskless return rate is .125, the current value of a security guaranteed to pay \$1 in one year would be \$1/1.125 = .88889. The security payoff vectors are as follows:

$$bu = \begin{bmatrix}20\\32\end{bmatrix}\ b = \begin{bmatrix}1\\1\end{bmatrix}\ c = \begin{bmatrix}4\\16\end{bmatrix}$$

Portfolio holdings are determined as follows:

$$\begin{bmatrix}20 & 1\\32 & 1\end{bmatrix}\cdot\begin{bmatrix}\gamma_{bu}\\\gamma_b\end{bmatrix}=\begin{bmatrix}4\\16\end{bmatrix}$$

The following includes the inverse matrix:

$$\begin{bmatrix} \gamma_{bu} \\ \gamma_b \end{bmatrix} = \begin{bmatrix} -.083333 & 0.083333 \\ 2.66667 & -1.66667 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 16 \end{bmatrix} = \begin{bmatrix} 1 \\ -16 \end{bmatrix}$$

We find that $\gamma_{bu} = 1$ and $\gamma_b = -16$. This implies that the payoff structure of a single call can be replicated with a portfolio comprising 1 share of stock and short-selling 16 T-Bills. This portfolio requires a net investment of 1×24-16×.88889 = $9.78. Since the call has the same payoff structure as this portfolio, its current value must be $9.78.

15.a. We will have a set of two payoff vectors in a two-outcome economy. The set is linearly independent. Hence, this set forms the basis for the two-outcome space. Since we have market prices for these two securities, we can price all other securities in this economy. First, we solve for the value of the $22-exercise price call as follows:

$$\begin{bmatrix} \gamma_s \\ \gamma_{c18} \end{bmatrix} = \begin{bmatrix} 15 & 0 \\ 25 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} \gamma_s \\ \gamma_{c18} \end{bmatrix} = \begin{bmatrix} .066667 & 0 \\ -.238095 & .142857 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ .42857 \end{bmatrix}$$

Thus, the call with an exercise price equal to $22 can be replicated with .42857 calls with an exercise price equal to $18. The value of this call equals .42857×7 = $3.

b. The riskless return rate is determined as follows:

$$\begin{bmatrix} .066667 & 0 \\ -.238095 & .142857 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} .066667 \\ -.095238 \end{bmatrix}$$

Since the riskless asset is replicated with .066667 shares of stock and short positions in .095238 calls, the value of the riskless asset is .066667×20-.095238×7 = .66667, implying a riskless return rate equal to 1/.66667 –1 = .50.

c. Solve for the value of the put as follows:

$$\begin{bmatrix} .066667 & 0 \\ -.238095 & .142857 \end{bmatrix} \begin{bmatrix} 25 \\ 15 \end{bmatrix} = \begin{bmatrix} 1.66667 \\ -3.8095 \end{bmatrix}$$

implying that its value is 1.66667 × 20 - 3.8095 × 7 = $6.67. Note that this put value is lower than either of the two potential cash flows that it may generate. This is due to the particularly high riskless return rate.

16.a. The one-year payoff vectors **j** for the stock, the T-bill (**b**) and the call (**c**) are given as follows:

$$\mathbf{j} = \begin{bmatrix} 34.722 \\ 50 \\ 72 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

b. At first glance, markets appear not to be complete in this 3-state economy because we know current market values for only 2 of the 3 securities. However, we do know that there are 4 outcomes in this recombining binomial framework: $\{u,u\}$, $\{u,d\}$, $\{d,u\}$and $\{d,d\}$, where $\{\{u,d\}, \{d,u\}\}$ combine and form a single state of nature. We know terminal payoffs for the bond and stock along with their market prices. And we do know that a binomial process generates stock prices, so that we really have only two potential outcomes at each time period, contingent on the prior outcome. This means that we can complete the market with only two securities if we were to that assume risk neutral probabilities are constant over the two 6-month intervals during the year. In effect, since each time period has two potential outcomes, and the risk neutral probabilities (and pure security prices) are the same in each time period, markets are complete.

c. The discount rate for each 6-month period is $d_1=\sqrt{.9}=.9487$. Thus, we have $\psi_{0,1;u}+\psi_{0,1;d}=.9487$ and $\psi_{0,1;u}=.9487-\psi_{0,1;d}$. Given the assumptions riskless returns are constant and that risk neutral probabilities are constant over the two 6-month intervals during the year, we can complete the market with only two securities, we calculate the pure security price associated with a downjump as follows:

$$50 = 34.722\psi_{0,1;d}{}^2 + 2\cdot 50\psi_{0,1;d}(.9487-\psi_{0,1;d}) + 72(.9487-\psi_{0,1;d})^2.$$

With a little algebra, we find that $\psi_{0,1;d}=.3775$. This means that the upjump pure security price is $\psi_{0,1;u}=.9487-.3775=.5712$.

d. Risk neutral probabilities for the first 6-month interval are $q_{0,1;d}=\psi_{0,1;d}/d_1=.3775/.9487=.3979$ and $q_{0,1;u}=\psi_{0,1;u}/d_1=.5712/.9487=.6021$.

e. The three pure security prices over the full 1-year period are $\psi_{0,2;d,d}=\psi_{0,1;d}{}^2=.3775^2=.143$, $\psi_{0,2;u\wedge d}=2\psi_{0.1;d}\psi_{0,1;u}=2\times.3775\times.5712=.431$ and $\psi_{0,2;u,u}=.5712^2=.326$.

f. Now, we value the call. If the stock price decreases in the first period to 41.667, the call will certainly produce a cash flow of zero in the second period and will be worth nothing. If the stock price increases in the first 6-month period by 20% to 60, the call will produce a cash flows of either zero or 12 at time 2, and will be worth (0×.3775) + (12×.5712) = 6.854 at time 1. Thus, at time zero, the call will be worth (0×.3775) + (6.854×.5712) = 3.915. Alternatively, multiplying our three pure security prices over the full year from part e by our call payoffs of 0, 0 and 12, respectively, and summing produces a call value of 3.915.