Chapter 3: Continuous Time and Continuous State Models

A. Continuous Time Payment Models

In this section, we begin with a simple model to value a zero-coupon bond in continuous time in an economy with a flat yield curve (market interest and discount rates are invariant with respect to bond maturity). We adjust this environment to allow for a finite number of changes in interest rates then continuous variations in rates over time to maturity for our zero-coupon bond.

Single Payment Model

In Chapter 2, we derived the continuous time present value for a single future payment:[1]

(1) $$PV = FVe^{-rT},$$

where r is the constant discount rate. Any asset making a single payment in the future can be valued with this model, as long as a continuous time model is appropriate for this purpose. For example, a T-year riskless zero-coupon bond with face value F is valued as $B_0 = PV_B = Fe^{-rT}$.

Pricing a Bond with a Continuous Deterministic Interest Rate

Since future interest rates are normally unknown, most models for interest rates are probabilistic in nature. Nevertheless, a deterministic model can serve as an excellent introduction to the more sophisticated models that we will cover extensively later in this text. In this section, we will assume that the interest rate $r(t)$ is a known real-valued continuous function of time t. The rate, while not stochastic, may or may not change with time.

The importance of Riemann Sums and their limits extends beyond their applications to finding areas under curves. Many continuous valuation models are derived by first approximating equations using discrete models resulting in Riemann Sums. Then, taking the limit as n approaches infinity, we derive an appropriate continuous valuation model. Here we will see this principle illustrated to price a bond.

As we reviewed in Chapter 2, the price of a zero-coupon bond is simply the present value of its face value at maturity. Let F denote the face value of the bond, and assume that it is purchased at time 0 and matures at time T. First, we approximate this continuous model with a discrete model in which interest is accumulated and compounded over a total of n equally intervals of time from 0 to T. This means that $\Delta t = (T-0)/n = T/n$. The n time intervals are $[t_{i-1},t_i]$ for $i=1,2,\ldots,n$ with $t_0=0$ and $t_n=T$. Consider the last time interval $[t_{n-1},t_n]$. The interest rate over this time period is $r(t_{n-1})$. Since $t_n-t_{n-1}=\Delta t$, then the amount of interest paid on 1 unit of money over the time interval $[t_{n-1},t_n]$ equals $r(t_{n-1})\Delta t$. If we were to regard time t_{n-1} to be the present time, then the present value of the bond equals $F(1+r(t_{n-1})\Delta t)^{-1}$. Next, consider the time interval $[t_{n-2},t_{n-1}]$. If we regard time t_{n-2} as the present time, and repeat the same argument, then the present value of the bond at time t_{n-2} equals $F(1+r(t_{n-2})\Delta t)^{-1}(1+r(t_{n-1})\Delta t)^{-1}$. Continuing this process all the way back to time 0 and choosing Δt to be very small, then we see that we can approximate the present value of the continuously modeled bond by the equation:

$$B_0 \approx F(1+r(t_0)\Delta t)^{-1}(1+r(t_1)\Delta t)^{-1}(1+r(t_2)\Delta t)^{-1}\cdots(1+r(t_n)\Delta t)^{-1}.$$

[1] Much of the material in this chapter was adapted from Knopf and Teall [2015].

Next, take the log of both sides of the equation above:

$$\ln B_0 \approx lnF - ln(1 + r(t_0)\Delta t) - ln(1 + r(t_1)\Delta t) - \cdots - ln(1 + r(t_n)\Delta t).$$

To estimate the log terms on the right-hand side of the equation, we find the second-order Taylor polynomial of the function $f(x) = ln(1+x)$ about $x = 0$. Taking derivatives, we calculate that: $f'(x) = (1+x)^{-1}$, $f''(x) = -(1+x)^{-2}$, and so $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$. The function $ln(1+x)$ can be estimated by its second-order Taylor polynomial about 0:

$$ln(1 + x) \approx 0 + (1)(x - 0) + \frac{1}{2}(-1)(x - 0)^2 = x - \frac{1}{2}x^2.$$

This can be used to approximate the log terms in the estimate for lnB_0:

$$ln(1 + r(t_{i-1})\Delta t) \approx r(t_{i-1})\Delta t - \frac{1}{2}[r(t_{i-1})]^2(\Delta t)^2.$$

As $\Delta t \to 0$, the term $\frac{1}{2}[r(t_{i-1})]^2(\Delta t)^2$ becomes negligible, resulting in the approximation:

$$ln(1 + r(t_{i-1})\Delta t) \approx r(t_{i-1})\Delta t.$$ [2]

Using this approximation in the estimate for lnB_0 gives:

$$ln\, B_0 \approx lnF - r(t_0)\Delta t - r(t_1)\Delta t - \cdots - r(t_{n-1})\Delta t = lnF - \sum_{i=1}^{n} r(t_{i-1})\Delta t.$$

Exponentiating both sides of the equation results in:

$$B_0 = Fe^{\left(-\sum_{i=1}^{n} r(t_{i-1})\Delta t\right)}.$$

In the limit, as n approaches infinity, we obtain the present value of the bond:

$$B_0 = Fe^{\left(\lim_{n\to\infty} -\sum_{i=1}^{n} r(t_{i-1})\Delta t\right)}.$$

Notice that on the right-hand side of the equation, the expression in the exponent is a limit of Riemann Sums with $t_i^* = t_{i-1}$. By the definition of the definite integral:

(3) $$B_0 = Fe^{\left(-\int_0^T r(t)dt\right)}.$$

Equation (3) gives the price of the bond.

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[2] Note: When Δt is very small $ln(1 + r(t_{i-1})\Delta t) \approx r(t_{i-1})\Delta t$. As $\Delta t \to 0$, this approximation improves, as long as r has a continuous second derivative.

Pricing a Bond with a Deterministic Continuous Rate: An Illustration

Here, we will find the price of a 10-year bond with a face value of $100 where the interest rate is given by *r(t) = .05 - .001t.* Notice that the interest rate declines proportionally over time in this simple model of the term structure. Using equation (3), we can easily find the price of the bond:

$$B_0 = 100e^{\left(-\int_0^{10}(.05-.001t)dt\right)} = 100e^{-\left(.05t-.0005t^2|_0^{10}\right)} = 100e^{-.45} = \$63.76$$

B. Differential Equations in Financial Modeling: An Introduction

In this and the next sections, we will examine riskless securities whose prices evolve continuously over time. We begin with simple growth models, while introducing the technique of solving differential equations by separating them. Then we target our discussion to assets such as stock, allowing for returns in continuous time. Nest, we will examine interest rates with a tendency to continuously revert towards some long-term rate. In later chapters, we will examine each of these processes under various types of uncertainty.

Financial economists and practitioners are often concerned with the development or change of a variable or asset over time. A *differential equation* can be structured to model the change (evolution or direction) of an asset's price over time. From this equation, a second equation (*solution*) might be derived to describe the asset's value (state or path) at a given point in time. The asset price is the dependent variable which is a function of the independent variable (time). More generally, one is interested in determining the solution for a dependent variable as a function of one or more independent variables. Often, the relationship between the dependent variable in terms of its independent variables is described by a differential equation. A differential equation is defined to be an equation that relates the dependent variable and one or more of its derivatives. A differential equation is defined to be an equation that relates the dependent variable and one of its derivatives. The solution to a differential equation is an explicit function that, when substituted for the dependent variable in the differential equation, leads to an identity. The following is a simple differential equation along with its solution involving dependent variable *x* and independent variable *t*:

(4) $$\frac{dx}{dt} = t$$

(5) $$x = \frac{1}{2}t^2 + C$$

where *C* is a constant. We verify the solution to differential equation 4 by noting that it represents the derivative of *x* with respect to *t* in its solution Equation 5. Equation 4 represents the change in variable *x* over time ($dx = t\,dt$). Note that this rate of change increases as *t* increases. Equation 5 represents the state or value of *x* at a given point in time *t*. Because equation 4 concerns only the first derivative of the function of *t*, it is referred to as a *first order differential equation*.

<u>Separable Differential Equations and Growth Models</u>

A differential equation is said to be *separable* if it can be rewritten in the form $g(x)dx = f(t)dt$. A separable differential equation written in this form can be solved by the following:

(6) $$\int g(x)dx = \int f(t)dt$$

The following is an example of a separable differential equation:

$$\frac{dx}{dt} = tx^2 - 2tx$$

We separate as follows:

$$\frac{dx}{dt} = t(x^2 - 2x)$$

$$\frac{dx}{(x^2-2x)} = tdt \text{ or } \frac{1}{(x^2-2x)}dx = tdt$$

$$\int \frac{1}{(x^2 - 2x)}dx = \int tdt$$

Growth Models

Consider the following example of a separable differential equation:

(7) $$\frac{dx}{dt} = rx$$

To solve this equation, we first separate the variables as follows:

$$\frac{1}{x}dx = rdt$$

Next, we integrate both sides and to obtain a *general solution* for *x*:

$$\int \frac{1}{x}dx = \int rdt$$

$$ln|\,x| + C_1 = rt + C_2$$

$$\ln|\,x| = rt + C_2 - C_1$$

When we integrate both sides of the equation, one obtains arbitrary constants $C_1$and C_2. Since these constants are arbitrary, we can define $C=C_1 - C_2$ which is still an arbitrary constant. Thus, whenever we integrate both sides of an equation, we only need to add an arbitrary constant to one side of the equation. So, we have in this case:

$$\ln\,|x| = rt + C.$$

$$e^{ln|\,x|} = e^{rt} \times e^C$$

$$|x| = e^{rt} \times e^C$$

$$x = \pm e^{C} e^{rt}$$

(8)
$$x = Ke^{rt} \ where \ K = \pm e^{C}$$

The constant *K* can assume any value. Thus, the general solution (or family of solutions) for our differential equation involves a constant that can assume any value. A *particular solution* results when *K* assumes a specific value. Stating the value of the function at a given moment of time is known as an *initial condition*. In our example, we have $x(0) = x_0$. The initial condition will determine the constant *K*, and we will then be able to write down the unique solution that satisfies both the differential equation and the initial condition. In this case, one particular solution for *x* would be $x = x_0e^{rt}$, where x_0 is the value of *x* when $t = 0$. In any case, this type of differential equation is typical of those used for modeling growth.

Security Returns in Continuous Time

Continuous time and continuous space models involve securities whose values evolve continuously over time (their prices can be observed at every instant) and can take on any real number value. Suppose that the evolution of a stock's price is modeled by the following separable differential equation:

(9)
$$\frac{dS_t}{dt} = \mu S_t$$

The term μ represents the security's drift or mean instantaneous rate of return. This differential equation is identical to equation (7) with $x = S_t$and $r = \mu$. By equation (8), the solution is:

(10)
$$S_t = Ke^{\mu t}$$

Equation 10 represents a general solution to our differential equation 9. If we set *K* equal to the stock's price S_0 at time zero (*K* can equal any constant), the particular solution to equation 9 would be:

(11)
$$S_t = S_0 e^{\mu t}$$

Differential equations such as (9) are useful in the modeling of security prices and are adaptable to the modeling of stochastic (random) return processes.

Illustration: Doubling an Investment Amount

Next, consider a security with value S_t in time *t* generating returns on a continuous basis such that the security's price doubles every 7 years. Suppose that the value of this security after 10 years were $50. What would have been the initial value S_0 of this security?

Equation (9) models this security price which can also be depicted by the security's return generating process:

$$\frac{dS_t}{S_t} = \mu dt$$

The solution to this equation is obtained with Equation (11). If we substitute $t = 7$ into the solution, we obtain:

$$S_7 = S_0 e^{7\mu} = 2S_0$$

Thus, $\mu = ln(2) \div 7 = .09902$. With this result, we can easily solve for the security's initial value:

$$S_0 = S_{10} e^{-1\ \times .09902} = \$50 e^{-10\times .09902} = \$18.58$$

Mean Reverting Interest Rates

While interest rates vary over time, they tend to be more likely to increase when they are "low" and decrease when they are high; that is, they tend to drift or revert to some long-term mean rate (long-term mean rate, not long-term rate). Suppose that μ represents the long-term mean interest rate, r_t the short-term rate at a particular time and λ an interest rate adjustment mechanism (also known as a "pullback factor:"

(12) $$dr = \lambda(\mu - r_t)dt$$

We will divide both sides by $(\mu - r_t)$ to separate and integrate:

$$\int \frac{dr}{(\mu - r_t)} = \int \lambda\, dt$$

Since $\mu\text{-}r_t = -\ (r_t - \mu)$, then:

$$\int \frac{1}{(r_t - \mu)} dr = -\int \lambda\, dt$$

$$\ln|r_t - \mu| = -\lambda t + C$$

$$|r_t - \mu| = e^{-\lambda t} e^{C}$$

(13) $$r_t - \mu = \pm e^{C} e^{-\lambda t}$$

Define the constant $K = \pm e^C$ so that K can be any constant (positive or negative). Substituting K into equation (13) and solving for r_t gives the solution:

(14) $$r_t = \mu + K e^{-\lambda}\ .$$

Suppose, for example, the interest rate 2 months ago was 18%, and currently is 16.5% as it drifts back to the long-term mean rate of 7%. How long will it take for the interest rate to drop below 10%?[3] First, we use the long-term drift to obtain $K = 11\%$:

[3] Note the similarity of this mean reverting interest structure to Newton's Law of Cooling. Also, note that the rate will never actually revert all the way back to its long-term mean.

$$r_0 = 18\% = 7\% + Ke^{-\lambda \times 0}.$$

Next, we solve for the "pullback factor" λ by using the interest rate drift over the past two months:

$$r_2 = 16.5\% = 7\% + 11\% \times e^{-2\lambda}$$

$$\frac{16.5\% - 7\%}{11\%} = e^{-2\lambda} = .86363636$$

$$\lambda = \frac{\ln(.86363636)}{-2} = .0733017$$

Finally, we solve for the time (in months) for the interest rate to drop below 10%:

$$r_t = 10\% = 7\% + 11\% \times e^{-.0733017t}$$

$$t = \frac{1}{-.0733017} ln \frac{3}{11} = 17.7251$$

We see now that it will take 17.7251 months from the start of the process for the interest rate to drift below 10%, or 15.7251 months from now.

C. Continuous State Models

In the previous chapter, we worked with securities in single-period economies with multiple states. Thus far in this chapter, we have worked only with single-state models that imply certainty. If only a single outcome is possible, the economy is certain. Multiple states imply uncertainty, but pricing is still possible in complete markets. In this section, we will seek to price securities relative to others in an economy with infinitely many states, though we will still rely on the same risk-neutral pricing methodology as in the previous chapter. First, we will start with a few preliminaries leading to a declaration of the distribution of states.

Option Pricing: The Elements

In this section, we will value options in single time-period frameworks with multiple potential outcomes in continuous outcome space. We will construct very simple probability distributions to compute expected values of options as functions of conditional expected stock values. Notice, following on our methodologies from Chapter 2, that we will work with risk-neutral q probabilities.

Expected Values of European Options

The expected future value of a European call is equal to its expected value conditional on its exercise at time T: ($E[(S_T - X)|S_T>X]$) multiplied by the probability that it will be exercised $P[(S_T>X)$. We will define $q[S_T]$ to be the hedging probability that $S_T = S$.[4] If the range of potential stock prices is continuous, this expected value is written as follows:

[4] A hedging probability, as discussed in Chapter 2, is the probability assigned by a risk neutral investor who values an asset at the prevailing market price.

$$E[c_T] = E[MAX[S_T - X), 0]] = \int_X^\infty (S_T - X) q[S_T] dS_T$$

where $q(S_T)$ is the density associated with a stock priced at S_T at time T. Note that the call's value is zero when $S_T<X$. The probability that the call will be exercised is:

$$P[S_T > X] = \int_X^\infty q[S_T] dS_T$$

The expected value of the stock and call given that the stock price exceeds the call exercise price are:

$$E[S_T | S_T > X] = \frac{\int_X^\infty S_T q[S_T] dS_T}{\int_X^\infty q[S_T] dS_T}$$

$$E[c_T | S_T > X] = \frac{E[c_T]}{P[S_T > X]} = \frac{\int_X^\infty (S_T - X) q[S_T] dS_T}{\int_X^\infty q[S_T] dS_T}$$

These conditional expected values do not account for stock prices when $S_T<X$. The expected value of the call is simply the product of its conditional expected value and the probability that it is exercised.

Call Options and Uniformly Distributed Stock Prices

Suppose that a stock's price at time $T$is expected to be uniformly distributed over the integer values that range from 1 to 100; that is:

$$q(x) = \begin{cases} \frac{1}{100}, & 1 \le x \le 100 \\ 0, & elsewhere \end{cases}$$

Further suppose that a call option with exercise price X = 60 trades on this stock with the following terminal payoff function:

$$c_T = MAX[S_T - X, 0]$$

The probability that the call will be exercised is computed as follows:

$$Q[S_T > 60] = \int_{60}^{100} \frac{1}{100} dS_T = \frac{1}{100} S_T \Big|_{60}^{100} = 1 - .6 = .4$$

The expected value of the stock, contingent on its price exceeding 60 is:

$$E[S_T|S_T > 60] = \frac{\int_{60}^{100} \frac{S_T}{100} dS_T}{\int_{60}^{100} \frac{1}{100} dS_T} = \frac{\frac{1}{200} S_T^2 \Big|_{60}^{100}}{.4} = \frac{32}{.4} = 80$$

The expected value of the call, contingent on it being exercised is:

$$E[c_T|S_T > X] = \frac{\int_X^{\infty} (S_T - X) q[S_T] dS_T}{\int_X^{\infty} q[S_T] dS_T} = \frac{\int_{60}^{100} \frac{S_T - 60}{100} dS_T}{\int_{60}^{100} \frac{1}{100} dS_T} = \frac{\frac{1}{200} S_T^2 - .6S_T \Big|_{60}^{100}}{.4}$$
$$= \frac{(50 - 60) - (18 - 36)}{.4} = \frac{8}{.4} = 20$$

The expected value of the call at time *T* is:

$$E[c_T] = \int_{60}^{100} (S_T - 60) \frac{1}{100} dS_T = \frac{1}{200} S_T^2 - .6S_T \Big|_{60}^{100} = (50 - 60) - (18 - 36) = 8$$

The present value of this call would simply be its discounted value:

$$c_0 = \int_X^{\infty} (S_T - X) q[S_T] dS_T \, e^{-rT} = 8e^{-r}$$

References

Knopf, Peter M. and John L. Teall (2015): *Risk Neutral Pricing and Financial Mathematics: A Primer*, Waltham, Massachusetts: Elsevier, Inc.

Exercises

1. Find the value of a bond that has a face value of $1,000 that matures in one year. The monthly rate at which interest accrues at time t (in months) after origination is determined by the following simple model: $r(t) = .007 - .00003t^2$.

2. Which of the following are separable differential equations?

a. $\frac{dy}{dt} = ty^2 - 2t^2y$

b. $\frac{dy}{dt} = ty \quad + t$

c. $\frac{dy}{dt} = y \quad + t$

3. Solve the following initial-value problem:

$$\frac{dy}{dt} = yt, \qquad y_0 = 100.$$

4.a. Solve the following equation assuming that $B_0 = 1$. This equation is used for pricing riskless bonds:

$$\frac{dB_t}{dt} = r_t B_t$$

b. Solve the above equation assuming that $B_4 = 1$.

5. Suppose that a statistical time series analysis revealed that an investment grows at a rate proportional to the square root of its current value with constant of proportionality k:

$$\frac{dV}{dt} = k\sqrt{V} \text{ and } V(0) = V_0.$$

Assume that V_0 is its initial value. Find the value of this investment as a function of time.

6. Suppose that the following separable differential equation reflects the continuous growth in a stock's price over time:

$$\frac{dS_t}{dt} = .01S$$

a. Solve this differential equation.

b. Suppose that $K = S_0 = 50$. Write the particular solution to this differential equation.

7. A T-year bond with face value $B_T = F$ has a price path given by the following differential equation:

$$\frac{dB_t}{B_t} = r_0 dt$$

Provide a general and a financially appropriate particular solution to this differential equation.

8. Suppose that a stock's price grows over time at rate μ as the firm produces profit, but the firm also pays a continuous dividend at a rate δ.

a. Write a differential equation to model the stock price over time.

b. What is the solution to this differential equation.

c. Suppose that $K = S_0 = 50$, $\mu = .05$ and $\delta = .02$. Write the particular solution to this differential equation.

9. Suppose that a particular process S_t satisfies the differential $dS_t = \mu(M - S_t)dt$ and initial value S_0 with $0 < S_0 < M$. Find the solution for S_t that is valid as long as $0 < S_t < M$.

10. Suppose a particular contract calls for accruals on an account to be credited such that over the time interval $[t, t + dt]$, where t is time in years, the amount that is credited equals $\$(10{,}000 + 500t)dt$. If these credits are discounted at an annual rate of 4%, what is the present value of the account after T years?

11. The People's Republic of Chrystal seeks to maintain a target exchange rate of μ = PRC6 relative to the U.S. dollar. Should the actual exchange rate r differ from its target rate, it will tend to drift towards the target rate as per the following function where λ and e^K are constants:

$$|\mu - r| = e^{\lambda t} e^K$$

Suppose the exchange rate 3days ago was 6.5 and is currently 6.4. How long will it take for the rate to drop below 6.1?

12. Suppose that short term interest rates follow the following mean-reverting process.:

$$dr = \lambda(\mu - r_t)dt$$

The long-term mean interest rate μ equals .06, the short-term rate r_0 currently equals .02, and the "pullback-factor" λ equals .4.

a. What will be the short-term rate r_1 in one year?
b. What will be the short-term rate r_2 in two years?

13. Here, we introduce the logistic growth function. Suppose that μ represents the long-term mean interest rate, r the rate at a particular time and λ an interest rate adjustment mechanism:

$$dr = \lambda r(\mu - r)dt$$

1. Is this differential equation separable? If so, demonstrate.
2. Solve this differential equation.
3. Suppose that the People's Republic of Chrystal seeks to maintain a target exchange rate of μ = PRC6 relative to the U.S. dollar, the adjustment mechanism λ in its logistic growth function is known to be .007 and the current exchange rate is 5. Find an equation that provides the exchange rate for any time t.
4. What is the exchange rate 10 days from now?

14. Suppose that a stock's price at time T is expected to be uniformly distributed over the range 10 to 20; that is:

$$q(x) = \begin{cases} \frac{1}{10}, & 10 \leq x \leq 20 \\ 0, & elsewhere \end{cases}$$

Further suppose that a put option with exercise price X = 16 trades on this stock with the following terminal payoff function:

$$p_T = MAX[X - S_T, 0]$$

a. Calculate the probability that the put will be exercised.
b. Calculate the expected value of the stock's price contingent on it exceeding the exercise

price of the put.

c. Calculate the conditional expected value of the put contingent on it being exercised.

d. Calculate the expected future value of the put.

e. If $T = 2$ and $r = .1$, what is the current value of the put? Assume that the stock price after the first period does not affect the parameters of the distribution of stock prices for the second period.

Solutions

1.In this problem, $F = 1000$, $T = 12$, and $r(t) = .007\text{-}.00003t^2$. Using equation (3), the price of the bond is:

$$B_0 = 1{,}000e^{\left(-\int_0^{12}(.007-.00003t^2)dt\right)} = 1{,}000e^{-\left(.007t-.00001t^3|_0^{12}\right)} = 1{,}000e^{-.06672} = \$935.46$$

2.a. No – We cannot separate y and t variables on two sides of the equation.

b. Yes - The right side factors as t(y + 1)

c. No - The right side cannot be factored as a product of a function of t and a function of y

3. First, separate by multiplying both sides of the differential equation by *dt* and divide both sides by *y*. Then integrate to obtain:

$$\int \frac{dy}{y} = \int tdt.$$

$$lny = \frac{1}{2}t^2 + C$$

since $y(0)=100>0$.

$$y = e^{\frac{1}{2}t^2+C} = Ke^{\frac{1}{2}t^2}.$$

The initial condition implies

$$100 = Ke^0 = K.$$

Thus

$$y = 100e^{\frac{1}{2}t^2}.$$

Alternatively, it is sometimes convenient to work out the solution by using a definite integral rather than an indefinite integral. So, we could also find the solution by the following steps:

$$\int_{100}^{y_T} \frac{dy}{y} = \int_0^T tdt.$$

Observe that to integrate in the variable *t* from 0 to T means that we are integrating in the variable *y* from $y(0) = 100$ to y_T.

$$ln\, y_T - ln \quad 100 = \frac{1}{2}T^2.$$

$$ln\left(\frac{y_T}{100}\right) = \frac{1}{2}T^2.$$

$$\frac{y_T}{100} = e^{\frac{1}{2}T^2}.$$

$$y_T = 100e^{\frac{1}{2}T^2}.$$

4.a. To solve this equation, we first separate the variables as follows:

$$\frac{1}{B_t}dB_t = r_t dt$$

Next we change the variable from *t* to *s* and integrate both sides from 0 to *t:*

$$\int_0^t \frac{1}{B_s}dB_s = \int_0^t r_s ds$$

$$ln\, B_t - ln B_0 = \int_0^t r_s ds$$

$$\ln\,(B_t\,/B_0) = \int_0^t r_s ds$$

$$e^{ln(B_t/B_0)} = e^{\int_0^t r_s ds}$$

$$B_t/B_0 = e^{\int_0^t r_s ds}$$

$$B_t = B_0 e^{\int_0^t r_s ds}.$$

Since $B_0 = 1$, then

$$B_t = e^{\int_0^t r_s ds}.$$

b. $1 = B_4 = B_0 e^{\int_0^4 r_s ds}$.

$$B_0 = e^{-\int_0^4 r_s ds}$$

$$B_t = e^{-\int_0^4 r_s ds} e^{\int_0^t r_s ds} = e^{-\int_0^4 r_s ds + \int_0^t r_s ds} = e^{\int_4^t r_s ds}$$

5. Let $V = V_t$ denote the value of the investment at time t. The problem states that

$$\frac{dV}{dt} = k\sqrt{V}\ and\ V(0) = V_0.$$

Multiplying both sides of the first equation by dt and dividing both sides by $\sqrt{V}$, we have:

$$\frac{dV}{\sqrt{V}} = kdt.$$

Next we integrate both sides of the equation from 0 to T:

$$\int_{V_0}^{V_T} \frac{dV}{\sqrt{V}} = \int_0^T kdt.$$

Evaluating the integrals, we see that

$$2\sqrt{V_T} - 2\sqrt{V_0} = kT.$$

Algebraically solving for V_T gives the solution to be

$$V_T = (\frac{kT}{2} + \sqrt{V_0})^2.$$

6.a. In this equation, $f(t) = .01$ and $g(S) = 1/S$, which separates as follows:

$$\tfrac{1}{S} dS = .01dt$$

$$g(S)dS = f(t)dt$$

We integrate both sides to obtain a *general solution* for S:

$$\int \frac{1}{S} dS = \int .01dt$$

$$\ln S = .01t + C$$

$$e^{\ln S} = e^{.01t} + e^C$$

$$S = Ke^{.01t}\ \ where\ K = e^C$$

b. A *particular solution* results when K assumes a specific value, say $K = S_0 = 50$. In this case, the particular solution for S could be $S_t = 50e^{.01t}$.

7. The solution to this differential equation gives the bond's price at time t will be obtained by

the following:

$$\int \frac{dB_t}{B_t} = \int r_0 dt$$

These integrals are solved as follows:

$$\ln B_t = r_0 t + C$$

We write the anti-logs of the results of both sides as:

$$e^{\ln \quad t} = e^{r_0 t + C}$$

$$B_t = K e^{r_0 t}$$

where $K = e^C$.This equation represents a general solution to our differential equation. Since $B_T = F$, evaluating the solution at $t = T$ gives: $F = K e^{r_0 T}$.
Solving for K we have:

$$K = F e^{-r_0 T}.$$

Substituting the value for K into the general solution for B_t we obtain the desired solution:

$$B_t = F e^{-r_0 T} e^{r_0 t} = F e^{-r_0 (T-t)}.$$

8.a. The differential equation is as follows:

$$\frac{dS}{dt} = (\mu - \delta) S_t$$

b. In this equation, f(t) = μ-δ and g(S) = 1/S, which separates as follows:

$$\frac{1}{S} dS = (\mu - \delta) dt$$

We integrate both sides to obtain a *general solution* for S:

$$\int \frac{1}{S} dS = \int (\mu - \delta) dt$$

$$\ln S = (\mu - \delta) t + C$$

$$e^{\ln S} = e^{(\mu - \delta) t} + e^C$$

$$S = K e^{(\mu - \delta) t} \ where \ K = e^C$$

c. A *particular solution* results when K assumes a specific value, say $K = S_0 = 50$. In this case, the particular solution for S could be $S_t = 50e^{(.05-.02)t}$.

9. First, divide both sides of the differential by $M - S_t$ to obtain:

$$\frac{dS_t}{M - S_t} = \mu dt.$$

Since, by the chain and log rules, the integral of $dS_t / (M - S_t)$ equals $-\ln(M - S_t)$, we will use the expression $\ln(M - S_t)$ to obtain the solution for S_t. The differential of the function $f(S_t) = \ln(M - S_t)$ is:

$$d[ln(M - S_t)] = \frac{-1}{M - S_t} dS_t = -\mu dt.$$

Changing the variable from t to s and integrating from 0 to t results in

$$ln(M - S_t) - ln(M - S_0) = -\mu t$$

or

$$ln\left(\frac{M - S_t}{M - S_0}\right) = -\mu t.$$

Exponentiating we have:

$$\frac{M - S_t}{M - S_0} = e^{-\mu t}.$$

Solving for S_t gives:

$$S_t = M - (M - S_0)e^{-\mu t}.$$

10. First, the present value of the fund over T years is:

$$PV[0,T] = \int_0^T (10{,}000 + 500t)e^{-.04t}dt.$$

We use the integration by parts formula:

$$\int_a^b udv = uv|_a^b - \int_a^b vdu.$$

For this problem choose

$$u = 10{,}000 + 500t \text{ } and \text{ } dv = e^{-.04t}dt,$$

so that

$$du = 500dt \text{ } and \text{ } v = \int e^{-.04t}dt = -25e^{-.04t}$$

Substituting these terms into the integration by parts formula gives

$$PV[0,T] = -25(10{,}000 + 500t)e^{-.04t} \quad |_0^T + \int_0^T 12{,}500e^{-.04t}\, dt$$

$$= -250{,}000e^{-.04} \quad + 12{,}500Te^{-.04T} + 250{,}000 - \frac{12{,}500}{.04}e^{-.04T} + \frac{12{,}500}{.04}$$

$$= -250{,}000e^{-.04T} + 12{,}500Te^{-.04} \quad + 250{,}000 - 312{,}500e^{-.04T} + 312{,}500$$

$$= -562{,}500e^{-.04} \quad + 12{,}500Te^{-.04T} + 562{,}500$$

$$= 562{,}500\left[1 - \left(1 + \frac{T}{45}\right)e^{-.04T}\right].$$

11. Since $r_0 = 6.5 > \mu = 6$, then $|\mu - r_t| = -(\mu - r_t) = r_t - \mu$. So the solution takes the form:

$$r_t = \mu + e^K e^{\lambda t}.$$

We evaluate the solution at $t = 0$ and use our initial condition in order to determine e^K:

$$r_0 = \mu + e^K e^{\lambda \times 0}.$$

$$6.5 = 6 + e^K$$

Thus, $e^K = .5$. Next, we solve for $\lambda = -0.074381184$ by using the exchange rate drift over the past 3 months:

$$r_3 = 6.4 = 6 + .5 \times e^{3\lambda}$$

$$\frac{6.4 - 6}{.5} = e^{3\lambda} = .8$$

Finally, we find that it takes t = 21.638 days from 3 days ago for the exchange rate to drop to 6.1:

$$r_t = 6.1 = 6 + .5 \times e^{-.07438t}$$

$$t = \frac{1}{-.07438} ln \frac{.1}{.5} = 21.638$$

12.a. First, we solve our differential equation as follows:

$$dr = \lambda(\mu - r_t)dt$$

$$\int \frac{dr}{(\mu - r_t)} = \int \lambda\, dt$$

$$\int \frac{1}{(\mu - r_t)} d(\mu - r_t) = -\int \lambda\, dt$$
$$ln|\mu - r_t| = -\lambda t + K$$
$$|\mu - r_t| = e^{-\lambda t} e^K$$
Since $\mu > r_0$, then $|\mu - r_t| = \mu - r_t$. Solving for r_t in the solution above gives:
$$r_t = \mu - e^K e^{\lambda t}.$$
We use the initial condition to determine e^K:
$$.02 = .06 - e^K,$$
so that $e^K = .04$. The particular solution for the short-term rate is:
$$r_t = .06 - .04e^{-.4t}.$$
The short-term rate in one year will be $r_1 = .06 - .04e^{-.4\times 1} = .03319$.

b. $r_2 = .06 - .04e^{-.4\times 2} = .042$.

13.a. The equation is separable. We will divide both sides by $r(\mu - r)$ to separate and integrate:
$$\int \frac{dr}{r(\mu - r)} = \int \lambda\, dt$$
b. Using partial fractions, this is written:
$$\int \left(\frac{1}{\mu r} + \frac{1}{\mu(\mu - r)}\right) dr = \int \lambda\, dt$$
It is easy to start with the equation immediately above, find a common denominator for the terms being added and verify that it equals the one above it in part a. For now, $0 < r < \mu$, leading to a general solution for the logistic equation:
$$\frac{1}{\mu} ln|r| - \frac{1}{\mu} ln|\mu - r| = \lambda t + K$$
$$\ln r - \ln(\mu - r) = \lambda\mu t + \mu K$$
$$e^{\ln r - \ln(\mu - r)} = e^{\lambda\mu t} e^{\mu K}$$
$$\frac{r}{(\mu - r)} = e^{\mu K} e^{\lambda\mu t}$$
$$r = (\mu - r) e^{\mu K} e^{\lambda\mu t}$$
$$r(1 + e^{\mu K} e^{\lambda\mu t}) = \mu e^{\mu K} e^{\lambda\mu t}$$
$$r = \frac{\mu e^{\mu K} e^{\lambda\mu t}}{(1 + e^{\mu K} e^{\lambda\mu t})}$$
c. Our solution to the logistic equation is:
$$r_t = \frac{6e^{\mu K} e^{.007\times 6t}}{(1 + e^{\mu K} e^{.007\times 1000t})}$$
The initial condition is:
$$r_0 = 5 = \frac{6e^{\mu K} e^{.007\times 6\times 0}}{(1 + e^{\mu K} e^{.007\times 6\times 0})} = \frac{6e^{\mu K}}{(1 + e^{\mu K})}$$
which implies that $e^{\mu K} = 5/(6-5) = 5$, which means that:
$$r_t = \frac{30e^{.007\times 6t}}{1 + 5e^{.007\times 6t}}$$
d. Simply substitute 10 for t in part c:
$$r_{10} = \frac{30e^{.007\times 6\times 10}}{1 + 5e^{.007\times 6\times 10}} = 5.30312$$

14.a. The probability that the put will be exercised is computed as follows:

$$P[S_T < 16] = \int_{10}^{16} \frac{1}{10} dS_T = \frac{1}{10} S_T \Big|_{10}^{16} = 1.6 - 1 = .6$$

b. The expected value of the stock, contingent on its price exceeding 16 is:

$$E[S_T | S_T > 16] = \frac{\int_{16}^{20} \frac{S_T}{10} dS_T}{\int_{16}^{20} \frac{1}{10} dS_T} = \frac{\frac{1}{20} S_T^2 \Big|_{16}^{20}}{.4} = \frac{20 - 12.8}{.4} = 18$$

c. The expected value of the put, contingent on it being exercised is:

$$E[p_T | \quad S_T < X] = \frac{\int_0^X (X - S_T) q[S_T] dS_T}{\int_0^X q[S_T] dS_T} = \frac{\int_{10}^{16} \frac{16 - S_T}{10} dS_T}{\int_{10}^{60} \frac{1}{10} dS_T} = \frac{1.6 S_T - \frac{1}{20} S_T^2 \Big|_{10}^{16}}{.6}$$

$$= \frac{(25.6 - 12.8) - (16 - 5)}{.6} = \frac{1.8}{.6} = 3$$

d. The expected value of the put at time T is:

$$E[p_T] = \int_{10}^{16} (16 - S_T) \frac{1}{10} dS_T = 1.6 S_T - \frac{1}{20} S_T^2 \Big|_{10}^{16} = (25.6 - 12.8) - (16 - 5) = 1.8$$

e. One important note in this problem is that the problem statement provides no reason to believe that the distribution of stock payoffs in the second period is any different from the distribution in the first period. Whether this might be a reasonable assumption might be a matter open for debate. We assume here that the parameters of the uniform distribution in the second period are identical to those of the first period, with the outcome of the first period not affecting the distribution in the second. However, waiting for two periods for the terminal cash flow will reduce the present value of the payoff. Thus, the present value of this put would simply be its discounted value:

$$c_0 = \int_X^\infty (S_T - X) q[S_T] dS_T \, e^{-rT} = 1.8 e^{-1.1 \times 2} = 1.4737$$