Chapter 7: Stochastic Processes: Introduction for Option Pricing

A. Random Walks and Martingales

In the previous two chapters, we priced securities in the marketplace as functions of pure security prices, synthetic probabilities, interest rates and discount functions. Here, we consider the functions that drive the random natures of one or more these values or functions. The mathematics in this Section A are presented rather formally, which will be more helpful later in this chapter when we begin to discuss Brownian Motion and Itô Processes, the stochastic processes that underly important continuous time pricing frameworks such as the Black Scholes model.

Stochastic Processes: A Brief Introduction

A *stochastic process* is a sequence of random variables X_t defined on a common probability space $(\Omega,\mathcal{F},\mathbb{P})$ and indexed by time t.[1] In other words, a stochastic process is a random series of values X_t sequenced over time. The values of $X_t(\omega)$ as t varies define one particular sample path of the process associated with the state or outcome $\omega \in \Omega$. The terms $X(\omega,t)$ and $X_t(\omega)$ are used synonymously here.

A *discrete time process* is defined for a countable set of time periods. This is distinguished from a *continuous time process* that is defined over an interval of the real line that consists of an infinite number of times. The *state space* X is the set of all possible values of the stochastic process $\{X_t\}$:

$$X = \{X_t(\omega) \text{ for some } \omega \in \Omega \text{ and some } t\}$$

The state space can be discrete (countable) or continuous. For example, if a bond price changes in increments of eighths or sixteenths, the state space for prices of the bond is said to be discrete. The state space for prices is continuous if prices can assume any real value.

The value of a real-world security is its value at any particular time, which depends on all past and present information. As we know, the set of possible events that determine the value of a random variable X_t is called its σ-algebra, which we will denote by $\mathcal{F}_t$. This motivates the following definitions: First, a *filtration* is a sequence of information sets indexed over time, so that each information set contains the required history for valuation at that time. More formally, a filtration is a sequence of σ-fields $\mathcal{F}_t$ such that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_t \subset \cdots \subset \mathcal{F}$. A sequence of *filtrations* $\mathcal{F}_t$ for a σ-algebra $\mathcal{F}$ is a sequence of σ-algebras with the property: [2]

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}$$

for discrete processes. Typically, the σ-algebras $\mathcal{F}_t$ are getting larger as they evolve in time t. Similarly, continuous processes will have the following property:

$$\bigcup_{s<t} \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$$

[1] See Section C in Chapter 2 for an introduction to probability spaces. Much of the content in this chapter is adapted from Knopf and Teall [2015].

[2] Filtrations can be characterized for our purposes as the information set required for valuation at any time or state.

for all $0 \le s \le t$.

A stochastic process X_t is said to be *adapted* to a filtration $\{\mathcal{F}_t\}$ if every measurable event for the random variable X_t is in the σ-algebra $\mathcal{F}_t$. Filtrations arise when securities are modeled by stochastic processes, because as time passes, the number of possibilities for the history of the security grows. In effect, a filtration represents the increasing stream of information (history) concerning the process.

Illustration: Filtrations in a Two time-period Random Walk

Suppose a stock has an initial price of X_0 at time 0, which, at time 1, will either increase or decrease by 1. Independently, the same potential price changes will occur at time 2. All of the possible distinct price 2-period change outcomes for the stock to time 2 are (1,1), (1,-1), (-1,1), and (-1,-1). For example, (1,-1) will mean the stock's price increased by 1 at time 1 and decreased by 1 at time 2. The sample space for this process is $\Omega = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$.

Actually, this stochastic process could have been defined for all times t = 0, 1, 2, 3, … . The example above is just for illustrative purposes. Suppose that the investor sells her stock at time 2, then keeps the proceeds in cash. In this scenario, for each time period t = 3, 4, …, the value of the stock $X_t = X_2$. This would then conform to the proper definition of a stochastic process. The σ-algebras $\mathcal{F}_t = \mathcal{F}_2$ for all t = 3, 4,… .

The price of the stock X_t for t = 0, 1 and 2 is adapted to a filtration that we will now describe. At time t = 0, the stock price is known. Since there has been no increase or decrease in the price at time 0, the time zero σ-algebra consists simply of:

$$\mathcal{F}_0 = \{\emptyset, \Omega\}.$$

At time 1, we have acquired information regarding whether the price increased or decreased by 1. At this time, the sets {(1,1), (1,-1)} and {(-1,1), (-1,-1)} are added to the σ-algebra such that the σ-algebra at time 1 becomes:

$$\mathcal{F}_1 = \{\emptyset, \{(1,1), (1,-1)\}, \{(-1,1), (-1,-1)\}, \Omega\}.$$

We cannot decouple the outcome (1,1) from the outcome (1,-1) in the σ-algebra $\mathcal{F}_1$ because we are unable to distinguish these two outcomes at time 1. That is, at time one, the outcome for time two has yet not occurred and is still unknown, therefore and not part of the time 1 information set.

At time 2, each of the separate outcomes (1,1), (1,-1), (-1,1), (-1,-1) are possible. Thus, $\mathcal{F}_2$ consists of the power set of $\Omega = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$:

$$\mathcal{F}_2 = \begin{matrix} \{\emptyset & \{(-1,-1)\} & \{(1,-1),(-1,1)\} & \{(1,1),(1,-1),(-1,-1)\} \\ \{(1,1)\} & \{(1,1),(1,-1)\} & \{(1,-1),(-1,-1)\} & \{(1,1),(-1,1),(-1,-1)\} \\ \{(1,-1)\} & \{(1,1),(-1,1)\} & \{(-1,1),(-1,-1)\} & \{(1,-1),(-1,1),(-1,-1)\} \\ \{(-1,1)\} & \{(1,1),(-1,-1)\} & \{(1,1),(1,-1),(-1,1)\} & \Omega\} \end{matrix}$$

Note that there are $2^4 = 16$ members of this σ-algebra. Observe that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$ and define $\mathcal{F} = \mathcal{F}_2$. Thus, $\{ \mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$is the filtration for this 2-period stock process.

Random Walks and Markov Processes

A *discrete Markov process*, often called a *discrete random walk*, is a non-continuous stochastic process in which the probability evolves to a given state at time t depends only on its immediately prior state at time t-1 and not on the remainder of its history. A Markov process is characterized by the condition

(1) $$P(X_t|X_{t-1}, X_{t-2}, \dots, X_0) = P(X_t|X_{t-1}).$$

The following provides an interpretation of this condition. Given the entire history of the process X_t from its start to time $t - 1$; namely, $X_0, X_1, \dots, X_{t-1}$, the probability that that the process will be in state X_t at the next moment time t depends only on its present state X_{t-1}. This property can be described as saying that a Markov Process is memoryless.

It can be proven that condition (1) above implies the more general condition:

(2) $$P(X_t|X_0, X_1, \dots, X_s) = P(X_t|X_s)\ \forall s < t.$$

This characterization of a Markov Process can be generalized further to the condition:

(3) $$P(X_t|\mathcal{F}_s) = P(X_t|X_s) \quad \forall s < t.$$

Recall that $\mathcal{F}_s$ is a σ-algebra of sets whose elements are drawn from all possible outcomes up to time s. Thus, if $\mathcal{F}_s$ is given, then we know which particular history of outcomes occurred to time s. This information determines the values of $X_s, X_{s-1}, \dots, X_0$, so that condition (3) implies condition (2). Condition (3) is in some sense the most natural of the three conditions in pricing securities in the market. That is, we typically wish to project some future price X_t based on some history to time s prior to time t. The σ-algebra $\mathcal{F}_s$ denotes all possible events that can occur in the market up to time s. Any security X_s that we wish to price in this market will be a random variable with $\mathcal{F}_s$ as its σ-algebra. These concepts will be illustrated shortly when we consider a family of Markov Processes modeling security prices.

Let X_t be a stochastic process over continuous time t, and random variables X_t be continuous. We define a *continuous Markov Process* by the condition:

(4) $$P(X_t \in A|\mathcal{F}_s) = P(X_t \in A|X_s)\ \forall s < t$$

where A is any measurable set on the real line. In equation (4), the parameters s, and t take on nonnegative real numbers. Note that it is not sufficient to define the probabilities for only single outcomes X_t as we did for the case when X_t is discrete, since the probability of any given single outcome is normally equal to 0 for continuous random variables X_t.

Illustration: The Random Walk

Consider the following type of random walk. [3] A person starts a walk at a certain integer position X_0 along an x-axis (…, -3, -2, -1, 0, 1, 2, 3, …) and at every moment (t = 0, 1, 2, …) , she chooses to take either one step to the right with probability p ($0 < p < 1$) or one step to the left

[3] Pearson [1905] described the optimal search process for finding a drunk left in the middle of a field. Left to stagger one step at a time in an entirely unpredictable fashion, she is more likely to be found where she was left than in any other position on the field.

with probability 1-p, such that each step to the right or left is independent of prior steps. Let Z_i be the random variable that equals 1 if the step is to the right at time i and -1 if the step is to the left at time i. It is obvious that if each step is taken independently of any previous steps, then $P(Z_s = a, Z_t = b) = P(Z_s = a)P(Z_t = b)$ when $s \neq t$. Thus, the random variables $\{Z_t\}$ are pairwise independent. Let X_t denote the position of the person on the x-axis after t steps have been taken. Notice that Z_t equal the increments (differences) $Z_t = X_t - X_{t-1}$ in the position of the person from time t-1 to time t. Thus, X_t can be expressed as $X_t = X_0 + Z_1 + Z_2 + ... + Z_t$. When the increments $\{Z_t\}$ are pairwise independent, we will prove below that:

$$P(X_t | X_0, X_1, \dots, X_{t-1}) = P(X_t | X_{t-1}).$$

This result is intuitively clear. If each step is independent of the previous steps, then the probability that the person will be at a certain position X_t at time t is only dependent upon her position X_{t-1} at the previous step. The rest of the past history of her walk is superfluous information. Thus, this random walk is a Markov Process. Random walks are of particular interest since they can serve as models of security prices.

Martingales and Submartingales

A *discrete martingale process* is a stochastic process X_t with the properties

1. $E[X_t | X_0, X_1, \dots, X_{t-1}] = X_{t-1}$
2. $E[|X_t|] < \infty$

for all $t = 1, 2, \dots$. In the first property above, regard $X_0, X_1, \dots, X_{t-1}$ as fixed (history) and X_t as a random variable. Thus, a martingale is a process whose future variations have no specific direction based on the process history ($X_0, X_1, \dots, X_{t-1}$). A martingale is said to be a "fair game" and will not exhibit consistent trends either up or down. The first property above implies the following:

(1') $$E[X_t | X_0, X_1, \dots, X_s] = X_s \ \forall s < t.$$

Since this property (1') is more general than property 1 above, we can use it to characterize the martingale instead in our discrete martingale definition.

In the case where X_t is a *continuous time martingale*, the second property must apply for all positive real numbers t, and the first property is replaced with $E[X_t \mid X_i, 0 \leq i \leq s] = X_s$ for all $s < t$ and all positive numbers t.[4] Observe that the definitions of a discrete time martingale and a continuous time martingale are equivalent except that the discrete case indices i, s, and t assume integer values, while the continuous case takes on all real number values. Analogous to our scenario involving Markov Processes, the first property of a martingale can be replaced with the more general condition:

[4] The second property is a technical condition that will be satisfied for every stochastic process that we study in this text. Thus, we will focus only on the first condition, excepting for Brownian motion, the most important continuous process in finance, in which case we will focus on both.

(5) $$E[X_t|\mathcal{F}_s] = X_s \; \forall s < t.$$

In each example involving a martingale in this book, the two expressions $E[X_t \mid X_i, 0 \le i \le s]$ and $[E[X_t|\mathcal{F}_s]$ will be equivalent. Thus we will use these notations interchangeably. Since $Z_t = X_t - X_{t-1}$, then for the discrete case we can also express the first martingale property above as follows:

$$E[X_t|X_0, Z_1, Z_2, \dots, Z_{t-1}] = X_{t-1}.$$

Consider the random walk described above with jumps equal to +1 with probability p and -1 with probability (1-p). Since $E[Z_t] = 1 \times p + (-1)(1-p) = 2p - 1$, $E[X_t \mid X_0, X_1, X_2, \dots, X_{t-1}] = E[X_t \mid X_{t-1}] = X_{t-1} + 2p - 1$. Thus, the random walk is a martingale when $p = 1/2$. Since $E[X_t \mid X_0, Z_1, Z_2, \dots, Z_{t-1}] = X_{t-1}$ when $p = 1/2$, a martingale's future has no specific direction in its trend from its present state. We also need to verify that the second condition $E[|X_t|] < \infty$ is satisfied. In this case, this is obvious since in time t, the farthest the random walk could have taken us would be t steps from the starting position at S_0. Thus, $E[|X_t|] \le MAX[|S_0 + t|, |S_0 - t|]$. Whenever we have a discrete process in which the change in the value of the process at each time increment is finite, then the second condition to be a martingale (submartingale or supermartingale to be covered next) is trivially satisfied.

Submartingales

A *submartingale* with respect to probability measure $\mathbb{P}$ is a stochastic process X_t in which the first property to be a martingale is replaced with:

$$E_{\mathbb{P}}[X_t \mid X_0, X_1, X_2, \dots, X_{t-1}] \ge X_{t-1}$$

A submartingale will tend either to trend upward over time or is a martingale. The definition of a *supermartingale* replaces the greater than or equal inequality above with a less than or equal inequality. A supermartingale will tend to trend downward over time or is a martingale.[5] In our random walk example above, X_t is a submartingale when $p \ge 1/2$ and is a supermartingale when $p \le 1/2$. Stock prices are often modeled as submartingales because they trend upwards due to the time value of money and investor risk aversion.

Equivalent Probabilities and Equivalent Martingale Measures

In Chapter 2, we used the concept of Arrow-Debreu (pure) securities and risk-neutral probabilities to begin to introduce and illustrate the concept of arbitrage-free pricing. Alternatively, physical probabilities are measures that we assign to outcomes that reflect the likelihoods of these outcomes actually occurring. Physical probabilities range between zero and one, they sum to one and their levels increase as presumed likelihoods of events increase. However, prices of assets need not be functions of these physical probabilities or the expected values based on physical probabilities. First, it might be perfectly reasonable to expect that asset prices will reflect investor preferences, heterogeneous expectations, risk aversion, lexicographic preference orderings, portfolios and other factors that will be unrelated to physical probabilities. For example, with heterogeneous investor expectations, prices could easily reflect one investor's probability estimates, but not another's. In fact, prices might not fully reflect any individual

[5] A submartingale has increments whose expected values equal or exceed zero; expected values of increments from a supermartingale are equal to or less than zero. Martingales are also both submartingales and supermartingales.

investor's expectations. In addition, as attractive as models based on physical probabilities might be, there is no market mechanism that forces prices to equal expected values based on physical probabilities.

On the other hand, risk neutral probabilities arise from the strongest of financial forces - arbitrage. Risk neutral probabilities are constructed from prices such that any violation of these prices or the risk neutral probabilities that they imply will create arbitrage opportunities. Thus, pricing relationships implied by appropriate functions of risk neutral probabilities must hold in the absence of arbitrage opportunities. Strictly speaking, from a mathematical perspective, risk neutral probabilities are probabilities in that they range from 0 to 1 and sum to 1 just as do physical probabilities, it is not necessary that they bear strong relationships to physical probabilities. Thus, risk-neutral probabilities do not strictly measure our opinions of likelihood in the way that physical probabilities do. However, they still resemble probabilities, they are essential to arbitrage-free pricing, and they can still be treated as probabilities for relative pricing purposes.

Numeraires

Thus far we have expressed all asset values in terms of dollars and returns relative to some currency or monetary unit such as dollars. Thus, the monetary unit (e.g., dollar) served as the *numeraire*, which is simply the unit in which values are expressed. However, we can just as easily express values in terms of other currencies, other securities such as pure securities as we did in Chapter 3, or riskless bonds as we will shortly. We can also express returns in terms of units of other securities such as forward contracts or even stocks. Flexibility in selecting the numeraire and an associated equivalent martingale measure affords us the ability of being able to use valuation techniques that otherwise would not be available or would be more complicated.

Equivalent Probability Measures

Recall that a probability measure $\mathbb{P}$ has the property that it is a mapping from an event space such that all events φ have probabilities $p[\varphi] \in [0, 1]$. An *equivalent probability measure* $\mathbb{Q}$ has the same null space as $\mathbb{P}$. That is, two probability measures are said to be equivalent ($\mathbb{P} \sim \mathbb{Q}$) if the set of events that have probability 0 under measure $\mathbb{P}$ (say, the physical probability measure) is the same as that set under the second measure $\mathbb{Q}$ (say, the risk-neutral probability measure). This also implies that $\mathbb{Q} << \mathbb{P}$ ($\mathbb{Q}$ is *absolutely continuous* with respect to $\mathbb{P}$, which means that $p(\varphi) = 0 \Rightarrow q(\varphi) = 0$) and $\mathbb{P} << \mathbb{Q}$ ($\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$; $q(\varphi) = 0 \Rightarrow p(\varphi) = 0$).[6] This implies that an equivalent probability measure is consistent with respect to which outcomes are possible. However, the actual non-zero probabilities assigned to events might differ.

Equivalent Martingale Measures

A probability measure $\mathbb{Q}$ is an *equivalent martingale measure* (also called a risk-neutral measure) to $\mathbb{P}$ in a complete market if $\mathbb{Q}$ and $\mathbb{P}$ are equivalent probability measures, and the price of every security in the market (using the riskless bond as the numeraire) is a martingale with respect to the probability measure $\mathbb{Q}$. We will examine an illustration of this later in the chapter. In a complete market in which there are no arbitrage opportunities, there will always exist a unique equivalent martingale measure. This measure can be used to obtain risk-

[6] If $\mathbb{Q} << \mathbb{P}$ and $\mathbb{P} << \mathbb{Q}$, then $\mathbb{Q} \sim \mathbb{P}$.

neutral pricing for every security in that market. There are a number of observations that relate to equivalent martingale measures:

1. For the finite outcome case, we will define a complete market to be one in which every security in the market can be expressed as a portfolio of a finite number of pure securities. The set of all possible securities in the complete market forms a vector space, and the set of pure securities of say *n* linearly independent vectors forms a basis for this vector space. Once we have determined the price for each pure security with respect to the equivalent martingale measure $\mathbb{Q}$, it is then a simple matter to find the price of any other security as a linear combination of the pure security prices (using the riskless bond as the numeraire).
2. Since the equivalent martingale measure $\mathbb{Q}$ is unique, one is free to choose any set of n linearly independent securities that form a basis for the market and that have already been priced in the market in order to construct the measure $\mathbb{Q}$. It may seem quite surprising that the measure $\mathbb{Q}$ turns out to be the same regardless of which basis of securities we choose to construct the measure $\mathbb{Q}$. This follows because of the linear relationship between the various securities' payoff and price vectors in a no-arbitrage market.
3. A market has a unique equivalent martingale measure if and only if it is both complete and arbitrage-free.
4. To use the riskless bond as the numeraire is equivalent to discounting the security by the riskless rate to obtain the present value of the security. The discounted security will be a martingale with respect to the measure $\mathbb{Q}$, and the risk-neutral price of the security is the expected value of the discounted security.
5. Since the discounted security price is a martingale, its expected value is consistent with the same return as the return on the riskless bond. If this were not the case, there would be an opportunity for arbitrage.

Pricing with Submartingales

We generally expect that financial securities requiring initial cash outlays will price as submartingales with respect to money because of the time value of money; investors demand compensation for giving up alternative uses of their initial investments:

$$E_{\mathbb{P}}[S_{t+\Delta t}] > S_t$$

However, if there exists an equivalent probability measure $\mathbb{Q}$ such that discounted security prices can be converted into martingales, this will ease the process of valuing derivative instruments relative to their underlying securities:

$$E_{\mathbb{Q}}\left[\frac{S_{t+\Delta t}}{(1+r)^{\Delta t}}\right] = S_t$$

We will discuss such conversions to martingales in this and the following chapters.

B. Binomial Processes: Characteristics and Modeling

A *Bernoulli trial* is a single random experiment with two possible outcomes (e.g., 0, 1, or *u* or *d*) whose outcomes depend on a probability. We will define a *binomial process* to be any stochastic process based on a series of *n* statistically independent Bernoulli (0,1) trials, all with the same outcome probability.

Binomial Processes

Consider the following Markov Process: $X_t = X_0 + aZ_1 + aZ_2 + \ldots + aZ_t$, where the random variable $Z_t(\omega) = 1$ if $\omega = u$ with probability p and $Z_t(\omega) = -1$ if $\omega = d$ with probability $1-p$ and a is a positive constant. We also assume that the random variables Z_t are pairwise independent. One can view this process as a random walk starting at X_0 and at each moment of time ($t = 1,2,\ldots$) taking a step of length a to the right with probability p and taking a step of length a to the left with probability $1 - p$. If it is the price of a security, then X_0 would be its initial value, and at each moment of time there is a probability p that it will increase in value by a and a probability $1 - p$ that it will decrease in value by a. Under the probability measure $\mathbb{P}$, the expected value of X_t at time t is:

$$E_{\mathbb{P}}[X_t] = E_{\mathbb{P}}[X_0 + aZ_1 + aZ_2 + \cdots + aZ_t] = X_0 + aE_{\mathbb{P}}[Z_1] + aE_{\mathbb{P}}[Z_2] + \cdots + aE_{\mathbb{P}}[Z_t]$$

$$= X_0 + a(2p - 1)t$$

since for each *i*, $E_{\mathbb{P}}[Z_i] = 1(p) + (-1)(1 - p) = 2p - 1$. Thus, the expected value of X_t depends linearly on time. This model is the stochastic version of a deterministic model in which an account pays a fixed and simple interest over time (when $p>1/2$). If one regards X_s as given, and calculates the expected value $E_{\mathbb{P}}[X_t|X_0, X_1, \ldots, X_s]$ for $s < t$, then similar to the calculation above:

$$E_{\mathbb{P}}[X_t|X_0, X_1, \ldots, X_s] = X_s + a(2p - 1)(t - s).$$

Observe that X_t is a martingale when $p = ½$, is a submartingale when $p > ½$, and is a supermartingale when $p < ½$. The process described here does not involve multiplicative factors or compounded returns. We will discuss multiplicative returns and compounding in the next subsection.

Binomial Returns Process

The binomial process described above can be applied to security prices, with prices increasing or decreasing by a specified monetary amount. However, this model does not provide for compounding of returns over time. For example, over a period of time, one might expect that a security with a high price to be subject to greater monetary fluctuation than a security with a low price; a $500 stock will probably experience greater price fluctuation than a $2 stock. Thus, it may be more realistic to instead construct a binomial process to security returns, that satisfies the following proportional model of returns:

$$\frac{S_t}{S_{t-1}} = (1 + aZ_t)$$

for $t = 1,2,\ldots$Thus, the security increases (upjump) by a factor $1 + a$ with probability p and decreases (downjump) by a factor $1 - a$ with probability $1 - p$ at each moment of time. This implies that

$$S_t = (1 + aZ_1)(1 + aZ_2) \cdots (1 + aZ_t)S_0.$$

From time 0 to time t, the security could have experienced a total of k upjumps for $k = 0,1,2,\ldots,t$. The number of upjumps is a binomial process and as we reviewed in Section 2.5.1. The probability that exactly k upjumps occurred from time zero to time t equals:

$$\binom{t}{k} p^k (1-p)^{t-k}.$$

Since k upjumps means the Z_i*'s* equaled one k times and equaled minus one $t - k$ times, this would result in the value of S_t equaling $(1+ a)^k(1 - a)^{t-k}S_0$. Thus, the expected value of S_t given S_0 is:

$$E_{\mathbb{P}}[S_t] = \sum_{k=0}^{t} (1+a)^k (1-a)^{t-k} S_0 \binom{t}{k} p^k (1-p)^{t-k}$$

$$= S_0 \sum_{k=0}^{t} \binom{t}{k} [p(1+a)]^k [(1-p)(1-a)]^{t-k}.$$

The binomial theorem states that

$$\sum_{k=0}^{t} \binom{t}{k} x^k y^{t-k} = (x+y)^t.$$

Choosing $x = p(1+ a)$ and $y = (1 - p)(1- a)$ in the equation above, gives the following result for the expected value:

$$E_{\mathbb{P}}[S_t] = S_0[p(1+a) + (1-p)(1-a)]^t = S_0[1 + a(2p-1)]^t.$$

In this case, the expected value has an exponential dependence on time. This model is the stochastic version of a deterministic model of an account paying compound interest or the return of a securty on a compound basis. By the same argument we used to obtain $E_{\mathbb{P}}[S_t]$, we can show

$$E_{\mathbb{P}}[S_t | S_0, S_1, \ldots, S_s] = S_s[1 + a(2p-1)]^{t-s}$$

for $s < t$. Clearly, $1 + a(2p - 1)$ equals 1 when $p = ½$, and S_t is a martingale. S_t is a submartingale when $p \geq ½$, and is a supermartingale when $p \leq ½$.

In the model above, we assumed that the price of the security increased by the multiplicative factor $1+a$ in the event of an upjump and decreased by the multiplicative factor $1 - a$ in the event of a downjump with $0 < a < 1$. This model can be generalized by allowing the multiplicative upward factor to be u for any $u > 1$ and the multiplicative downward factor to be d for any $0 < d < 1$. The symbols u and d can simply refer to an upjump or downjump; u and d can also refer to particular values for multiplicative jumps. As in the previous model, we assume that the probability of an upjump or downjump at a particular time *t* is independent of the probability of an upjump or downjump at any other time *s*. Similar to the calculation earlier, the expected value of the security is:

$$E_{\mathbb{P}}[S_t] = S_0[pu + (1-p)d]^t.$$

Since the security price follows the same probability and pricing law in this model if we shift the time by any value *s*, then:

$$E_{\mathbb{P}}[S_t|S_s] = S_s[pu + (1-p)d]^{t-s}$$

for $s < t$. The process S_t is a martingale as long as $pu + (1-p)d = 1$, or $p = (1 - d)/(u - d)$.

Illustration: Binomial Outcome and Event Spaces

In this section, we will consider a relatively simple 2-time period time binomial process in order to illustrate the construction of risk-neutral pricing for a security. Table 1 depicts a sampling of $n = 2$ successive independent and identically distributed jumps, in which each Bernoulli trial can result in one of two potential outcomes. One such sampling might be based on a stock whose price can either rise or fall in each of 2 sequential transactions. As before, the letter *u* (upjump) will mean the stock increased in value at time t, and the letter *d* (downjump) will mean the stock price decreased. For example, the letters *ud* means that the price of the stock went from 10 to 15 at time 1 and then from 15 to 7.5 at time 2. Suppose that the *physical probability* (as opposed to risk neutral probability) associated with a stock price increase in each transaction is given to be *p*, implying a probability of (1-*p*) for a price decline in each transaction. Figure 1 and the listing below depict first the sample space Ω and then the filtration:

$$\Omega = \{uu, ud, du, dd\}$$

$$\mathcal{F}_0 = \{\varnothing, \Omega\}$$

$$\mathcal{F}_1 = \{\varnothing, \{uu, ud\}, \{du, dd\}, \Omega\}$$

$$\mathcal{F}_2 = \{\varnothing, \{\text{uu}\}, \{\text{ud}\}, \{\text{du}\}, \{\text{dd}\}, \{\text{uu, ud}\}, \{\text{uu, du}\}, \{\text{uu, dd}\}, \{\text{ud, du}\}, \{\text{uu, dd}\}, \{\text{du, dd}\},$$

$$\{\text{uu, ud, du}\}, \{uu, ud, dd\}, \{uu, du, dd\}, \{ud, du, dd\}, \Omega\}.$$

Notice in Figure 1 that the process is *recombining* (multiple paths can lead to the same time 2 price).

Pure Security Price	**At Time**	**Maturity**	**Time 1 Outcome**	**Time 2 Outcome**	**Numerical Value**
Spot Prices					
$\psi_{0,1;u}$	0	1	*upjump*	N/A	.6
$\psi_{0,1;d}$	0	1	*downjump*	N/A	.2
$\psi_{0,2;u,u}$	0	2	*upjump*	*upjump*	.36
$\psi_{0,2;u,d}$	0	2	*upjump*	*downjump*	.12
$\psi_{0,2;d,u}$	0	2	*downjump*	*upjump*	.12
$\psi_{0,2;d,d}$	0	2	*downjump*	*downjump*	.04
Forward Prices					
$\psi_{1,2;u,u}$	1	2	*upjump*	*upjump*	.6
$\psi_{1,2;u,d}$	1	2	*upjump*	*downjump*	.2
$\psi_{1,2;d,u}$	1	2	*downjump*	*upjump*	.6
$\psi_{1,2;d,d}$	1	2	*downjump*	*downjump*	.2

This table gives pure security prices at time *t* (second column) for instruments that pay off (mature) in *T* years (third column) contingent on the outcome in the fourth column and following the jump listed in the fifth column. The final column lists the pure security prices. The subscripting for 1-period spot prices (first subscript is zero) identifies the contract as follows: time zero (first subscript) pure security price paying 1 at time 1 (second subscript) contingent on up/downjump (third subscript). The subscripting for 2-period spot prices (first subscript is zero) identifies the contract as follows: pure security price at time 0 (first subscript) paying 1 at time 2 (second subscript) contingent on up/downjump at time 1(third subscript) and on up/downjump at time 2 (fourth subscript). The subscripting for forward prices (first subscript exceeds zero) identifies the contract as follows: pure security price at time 1 (first subscript) paying 1 at time 2 (second subscript) contingent on up/downjump at time 1(third subscript) and on up/downjump at time 2 (fourth subscript).

Table 1: Pure Security Prices

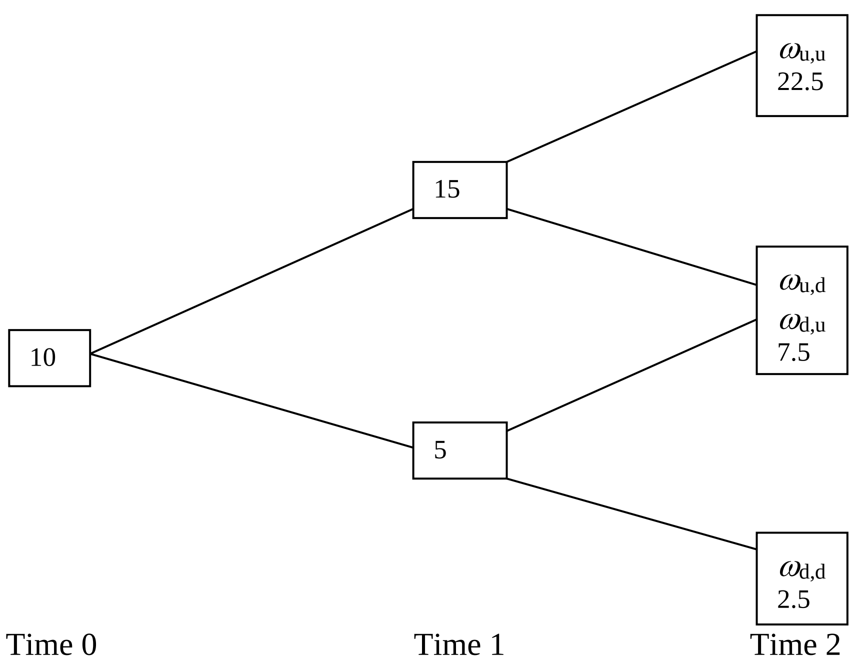

Figure 1: Two Time Period Binomial Model - Current and Potential Stock Prices

Suppose that a stock (non-dividend paying) trades in this economy, such that its current and expected prices are given in Figure 1. In addition, three riskless bonds trade in this economy, all of which have face values equal to 1. The first two bonds originate in time 0, mature in years 1 and 2, respectively, and currently trade for .8 and .64. The third bond will originate in time 1, and in the absence of arbitrage opportunities, its forward contract must trade with a settlement price equal to .8 based on known prices of the first two bonds.

Pure Security Prices

Next, we will discuss Arrow-Debreu (pure) security prices, one associated with each starting and stopping node combination in our 2-time period model (See Table 1 below). First, we will list spot (time 0) prices for investments extending from time 0 to time 1. Let $\psi_{0,1;u}$ be the time zero price of a pure security that is worth 1at time 1 if and only if an upjump occurs at time 1. The first subscript 0 refers to the initial time 0, the second subscript 1 refers to the next time 1, and the third subscript *u* refers to the fact that there is an upjump (at time 1). Let $\psi_{0,1;d}$ be the time zero price of a pure security that is worth 1at time 1 if and only if a downjump occurs at time 1. The third subscript *d* in this case refers to the fact that there is a downjump (at time 1). Subscripts before the semicolon refer to time periods, and subscripts after the semicolon refer to the occurrence of upjumps or downjumps. The vector $[1 \quad 0]^T$ will denote the payoff of the pure security identified with an upjump occurring at time 1. The vector $[0 \quad 1]^T$ will denote the payoff of the pure security identified with a downjump occurring at time 1. To purchase the stock at time 0 at a price of 10 and hold to it until time 1 means that at time 1 the portfolio of the owner takes the form of the vector $[15 \quad 5]^T$ because it will pay 15 if an upjump occurs and will pay 5 if a downjump occurs. Since an investor is willing to pay $\psi_{0,1;u}$ at time 0 for a payoff vector $[1 \quad 0]^T$ and pay $\psi_{0,1;d}$ at time 0 for a payoff vector $[0 \quad 1]^T$ and since the value of the stock portfolio $[15 \quad 5]^T$ is equal to 10 at time 0, then:

(6) $$15\psi_{0,1;u} + 5\psi_{0,1;d} = [15 \quad 5]\begin{bmatrix}\psi_{0,1;u}\\ \psi_{0,1;d}\end{bmatrix} = 10$$

Also, if an investor has the riskless portfolio $[1 \quad 1]^T$, then she is guaranteed to be paid 1 at time 1. As this is equivalent to holding the bond that pays 1 at time 1, and since the time 0 price of the portfolio $[1 \quad 1]^T$ is .8, then:

(7) $$\psi_{0,1;u} + \psi_{0,1;d} = [1 \quad 1]\begin{bmatrix}\psi_{0,1;u}\\ \psi_{0,1;d}\end{bmatrix} = .8$$

Combining these two vector equations into one matrix equation yields:

(8) $$\begin{bmatrix}15 & 5\\ 1 & 1\end{bmatrix}\begin{bmatrix}\psi_{0,1;u}\\ \psi_{0,1;d}\end{bmatrix} = \begin{bmatrix}10\\ .8\end{bmatrix}$$

since we can invert the 2×2 matrix above and rewrite (8) as follows:

$$\begin{bmatrix}0.1 & -0.5\\ -0.1 & 1.5\end{bmatrix}\begin{bmatrix}10\\ .8\end{bmatrix} = \begin{bmatrix}\psi_{0,1;u}\\ \psi_{0,1;d}\end{bmatrix} = \begin{bmatrix}0.6\\ 0.2\end{bmatrix}$$

The solution is $\psi_{0,1;u}$ =.6 and $\psi_{0,1;d}$=.2, and the results are listed in Table 1 above.

Before we discuss the 2-period spot prices, we also have time 1 forward prices for pure securities that pay off in time 2. As described in Table 1, let $\psi_{1,2;u,u}$ be the time 1 price of a pure security that pays 1 at time 2 if and only if upjumps occur at both times 1 and 2 (Outcome $\omega_{u,u}$). The remaining forward prices $\psi_{1,2;u,d}$, $\psi_{1,2;d,u}$, and $\psi_{1,2;d,d}$ are defined similarly. Note that two paths lead to a time 2 price of 7.5. The same procedure can used to calculate pure security prices as we did going from time 0 to 1. These forward pure security prices are calculated from the following and listed in Table1:

(9) $$\begin{bmatrix} 22.5 & 7.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \psi_{1,2;u,u} \\ \psi_{1,2;u,d} \end{bmatrix} = \begin{bmatrix} 15 \\ .8 \end{bmatrix} \;;\quad \begin{matrix} \psi_{1,2;u,u} = .6 \\ \psi_{1,2;u,d} = .2 \end{matrix}$$

(10) $$\begin{bmatrix} 7.5 & 2.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \psi_{1,2,d,u} \\ \psi_{1,2;d,d} \end{bmatrix} = \begin{bmatrix} 5 \\ .8 \end{bmatrix} \;;\quad \begin{matrix} \psi_{1,2;d,u} = .6 \\ \psi_{1,2;d,d} = .2 \end{matrix}$$

We can calculate the 2-period pure security spot prices from the 1-period spot and forward prices. Again as described in Table 1, let $\psi_{0,2;u,u}$ be the time zero price of a pure security that pays 1at time 2 if and only if there are consecutive upjumps (denoted by the outcome $\omega_{u,u}$). Thus, the time zero pure security price associated with time 2 outcome $\omega_{u,u}$ is $\psi_{0,2;u,u}$. The remaining 2-period spot prices $\psi_{0,2;u,d}$, $\psi_{0,2;d,u}$, and $\psi_{0,2;d,d}$ are defined similarly. Internal pricing consistency (no-arbitrage pricing) requires that $\psi_{0,2;u,u} = \psi_{0,1;u}\times\psi_{1,2;u,u} = .6 \times .6 = .36$ and $\psi_{0,2;d,d} = \psi_{0,1;d}\times\psi_{1,2;d,d} = .2 \times .2 = .04$. Also, no-arbitrage pricing requires that $\psi_{0,2;u,d} = \psi_{0,1;u}\times\psi_{1,2;u,d} = .6 \times .2 = .12$ and $\psi_{0,2;d,u} = \psi_{0,1;d}\times\psi_{1,2;d,u} = .2 \times .6 = .12$. For the recombining part of our lattice, the pure security price for the set of outcomes $\{\omega_{u,d}, \omega_{d,u}\}$ is $\psi_{0,2;u,d} + \psi_{0,2;d,u} = .12+.12 = .24$. These no-arbitrage relationships are captured in the following pure security prices matrix equation and listed in Table 1 above:

(11) $$\begin{bmatrix} \psi_{0,2;u,u} \\ (\psi_{0,2;u,d} + \psi_{0,2;d,u}) \\ \psi_{0,2;d,d} \end{bmatrix} = \begin{bmatrix} \psi_{1,2;u,u} & 0 \\ \psi_{1,2;u,d} & \psi_{1,2;d.u} \\ 0 & \psi_{1,2;d,d} \end{bmatrix} \begin{bmatrix} \psi_{0,1;u} \\ \psi_{0,1;d} \end{bmatrix}$$

$$\begin{bmatrix} \psi_{0,2;u,u} \\ (\psi_{0,2;u,d} + \psi_{0,2;d,u}) \\ \psi_{0,2;d,d} \end{bmatrix} = \begin{bmatrix} .6 & 0 \\ .2 & .6 \\ 0 & .2 \end{bmatrix} \begin{bmatrix} .6 \\ .2 \end{bmatrix} = \begin{bmatrix} .36 \\ .24 \\ .04 \end{bmatrix}$$

Our stock and 2-year bond can be priced from 2-year payoff vectors as follows:

$$\begin{bmatrix} 22.5 & 7.5 & 2.5 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \psi_{0,2;u,u} \\ (\psi_{0,2;u,d} + \psi_{0,2;d,u}) \\ \psi_{0,2;d,d} \end{bmatrix} = \begin{bmatrix} S_0 \\ B_0 \end{bmatrix}$$

$$\begin{bmatrix} 22.5 & 7.5 & 2.5 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} .36 \\ (.12 + .12)) \\ .04 \end{bmatrix} = \begin{bmatrix} 10 \\ .64 \end{bmatrix}$$

Bond Prices

From these 9 pure security prices, we can obtain spot prices for 1- and 2-year zero-coupon bonds with face value 1 along with discount functions and their associated rates as follows:[7]

$$B_{0,1} = d_{0,1} = \psi_{0,1;u} + \psi_{0,1;d} = .8; \quad r_{0,1} = \frac{1}{\psi_{0,1;u} + \psi_{0,1;d}} - 1 = .25$$

$$B_{0,2} = d_{0,2} = \psi_{0,2;u,u} + \psi_{0,2;u,d} + \psi_{0,2;d,u} + \psi_{0,2;d,d} = .64$$

$$r_{0,2} = \sqrt{\frac{1}{\psi_{0,2;u,u} + \psi_{0,2;u,d} + \psi_{0,2;d,u} + \psi_{0,2;d,d}}} - 1 = .25$$

We can also obtain forward bond prices, discount functions and rates contingent on realization of outcomes *u* and *d*, respectively, at time 1:

$$B_{1,2;u} = d_{1,2;u} = \psi_{1,2;u,u} + \psi_{1,2;u,d} = .8\,; \quad r_{1,2;u} = \frac{1}{\psi_{1,2;u,u} + \psi_{1,2;u,d}} - 1 = .25$$

$$B_{1,2;d} = d_{1,2;d} = \psi_{1,2;d,u} + \psi_{1,2;d,d} = .8; \quad r_{1,2;d} = \frac{1}{\psi_{1,2;d,u} + \psi_{1,2;d,d}} - 1 = .25$$

Physical Probabilities, The Equivalent Martingale Measure and Change of Numeraire

Physical Probabilities

Suppose that in our process, we define the physical probability of each upjump to be p_u = .8; the physical probability of each downjump is p_d = .2. For our process, we will define $p_0(\omega_t)$ to be the time-zero physical probability of a time *t* outcome ω_t in Ω. For example, at time 0, $p_0(u)$ = p_u =.8 and $p_0(d) = p_d$ = .2. Because of independence, at time 0, $p_0(u, u) = p_u^2$ = .64, $p_0(u, d)$ + $p_0(d, u)$= $2p_u p_d$ = .32 and $p_0(d, d) = p_d^2$ = .04. These are all non-zero (relevant) physical probabilities in probability measure ℙ, and are listed in Table 2 below. However, now that we have set forth these physical probabilities, we will not actually have much use for them for valuation purposes. Again, physical probabilities, however much we might want to calculate them, are pretty useless here. In fact, these physical probability estimates might vary among individuals, might be unrelated to actual market prices and really are not enforced by any particular market discipline. By converting to risk-neutral probability measure ℚ, we will be able to conduct meaningful risk-neutral pricing of the stock, options and other securities. We will compute risk-neutral probabilities from pure security prices and will represent the stock as a martingale with respect to the risk-neutral probability measure.

[7] Alternatively, we might assume continuous compounding of interest such that $r_{t,T} = -\ln \sum \psi_{0,t,i}$.

Risk Neutral Probability	At Time	Maturity	Outcome	Physical Probability
$q_{0,1;u}$=.75	0	1	u	.8
$q_{0,1;d}$ =.25	0	1	d	.2
$q_{0,2;u,u}$=. 5625	0	2	u,u	.64
$q_{0,2;u,d}$ =.1875	0	2	u,d	.16
$q_{0,2;d,u}$ =.1875	0	2	d,u	.16
$q_{0,2;d,d}$ =.0625	0	2	d,d	.04
$q_{1,2;u,u}$ =.75	1	2	u,u	.8
$q_{1,2;u,d}$ =.25	1	2	u,d	.2
$q_{1,2;d,u}$ =.75	1	2	d,u	.8
$q_{1,2;d,d}$ =.25	1	2	d,d	.2

This table gives risk-neutral (hedging) probabilities in the first column at time t (second column) for instruments that pay off (mature) in *T* years (third column) contingent on the outcome in the fourth column and following the event listed in the fifth column. The final column lists the numerical values for physical probabilities.

Table 2: Risk Neutral Probabilities

The Equivalent Martingale Measure

Next, we will characterize the set of risk-neutral probabilities or the *equivalent martingale measure* $\mathbb{Q}$ for this space. We obtain the time *t* risk-neutral probability for outcome *i* at time *T* by dividing the pure security price $\psi_{t,T;i}$ by the time (*t*, *T*) riskless discount function $d_{t,T}$:

$$q_{t,T;i} = \psi_{t,T;i}/d_{t,T} = \psi_{t,T;i}/\mathrm{B}_{t,T} \tag{12}$$

In Chapter 3, we learned that:

$$q_{t,T;i} = \frac{\psi_{t,T;i}}{\sum_{j=1}^{n} \psi_{t,T;j}},$$

but these results are equivalent since $\mathrm{B}_{t,T} = \sum_{j=1}^{n} \psi_{t,T;j}$. As you can see from equation (10), the risk-neutral probability is the price of the pure security associated with that time period and outcome in terms of the riskless bond serving as the numeraire. Thus, the numeraire is the price of the one-dollar face value bond maturing in the time period from *t* to T. These risk neutral probabilities are all listed in Table 2 and they can be obtained from Table 1. For example, $q_{0,1;u}$= $\psi_{0,1;u}/B_{0,1}$=.6/.8=.75, $q_{1,2;u,d}$= $\psi_{1,2;u,d}/B_{1,2}$=.2/.8=.25, and $q_{0,2;d,d}$= $\psi_{0,2;d,d}/B_{0,2}$=.04/.64=.0625.

Even though the risk neutral probabilities in Table 2 differ substantially from the physical probabilities listed above, the two sets of four probability measures are equivalent. This is simply because at each node on our lattice, we see that each non-zero probability in probability measure $\mathbb{P}$ corresponds to a non-zero probability in probability measure $\mathbb{Q}$. Thus, our risk-neutral probability measure $\mathbb{Q}$ is equivalent to our original physical probability measure $\mathbb{P}$. Next, we will demonstrate that the stock price is a martingale with respect to our risk neutral probability measure $\mathbb{Q}$ using our alternative numeraire, the riskless bond.

Change of Numeraire and Martingales

To express the stock price as a martingale, we will change the numeraire for valuing our securities. Rather than express security values in terms of monetary units (e.g., dollars), we will express values in terms of some security, namely, the riskless bond. Notice that at time zero, the value of a single share of stock S_0, 10, is S_0/B_0=10/.8=12.5 times the value of the bond, B_0 = .8. We also point out that the bond's value at time zero in bond units is 1 since $B_0/B_0=1$. Thus, the value of 1 share of stock at time zero equals that of 12.5 bonds. In addition, notice that with our equivalent probability measure $\mathbb{Q}$, the time zero expected future value of the share at time 1 given the time zero value of the share of stock is also 12.5 times that of the bond:

$$\begin{bmatrix} 15 & 5 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} q_{0,1;u} \\ q_{0,1;d} \end{bmatrix} = \begin{bmatrix} 12.5 \\ 1 \end{bmatrix} \;;\quad \begin{matrix} q_{0,1;u} = .75 \\ q_{0,1;d} = .25 \end{matrix}$$

$$E_{\mathbb{Q}}[S_{0,1}|S_0/B_0] = 15q_{0,1;u} + 5q_{0,1;d} = 12.5 = S_0/B_0.$$

Since the time-zero expected value of the stock at time 1 (given the time-zero stock value in bond numeraire) under the probability measure $\mathbb{Q}$ equals the time-zero stock value (in bond numeraire), then the stock price has the martingale property at time 1. We can demonstrate similar martingale properties going from time 1 to time 2, and from time 0 to time 2. This will be explained in more detail in the next section. Thus, in this two time period process, the probability measure $\mathbb{Q}$ is an equivalent martingale measure to $\mathbb{P}$.

Also note that if we convert from the bond numeraire back to the original currency (such as dollars), we get the right values at time 0. The time-zero value of the stock is 12.5 times that of the bond. The bond is worth .8 dollars at time zero, therefore the stock has a time zero value equal to 10 dollars:

$$S_0 = \frac{E_{\mathbb{Q}}[S_{0,1}]}{(1+r)^{\Delta t}} = \frac{15q_{0,1;u}+5q_{0,1;d}}{1+.25} = 10$$

$$B_0 = \frac{E_{\mathbb{Q}}[B_{0,1}]}{(1+r)^{\Delta t}} = \frac{1q_{0,1;u}+1q_{0,1;d}}{1+.25} = .8$$

<u>Binomial Pricing, Change of Numeraire and Martingales</u>

Here, we will prove that for an arbitrary risk-free binomial pricing model, the stock price S_t is a martingale with respect to the price of a bond (the riskless bond will be the numeraire). We will price the stock and bond at each time period given their potential subsequent cash flows. After each time period, we will verify that the stock price, when priced with the bond as the numeraire, satisfies the martingale property.

Pricing the Stock and Bond from time 0 to time 1:

Let $S_{0,1;u}$ and $S_{0,1;d}$ denote the time 1 prices of the stock after an upjump and downjump, respectively, based on the price path from time 0. $\psi_{0,1;u}$ and $\psi_{0,1;d}$ will denote the prices that an investor is willing to pay at time 0 for pure securities that pay off 1 at time 1 if and only if the stock has an upjump (respectively downjump) to the value $S_{0,1;u}$ (respectively $S_{0,1;d}$) from time 0 to time 1. Let $B_{0,1}$ be the time zero price of a bond that pays

1 at time 1. To purchase the stock at time 0 at a price of S_0 and hold to it until time 1 means that at time 1 the portfolio of the owner takes the form of the vector $[S_{0,1;u} \quad S_{0,1;d}]^T$ because it will pay $S_{0,1;u}$ if an upjump occurs and will pay $S_{0,1;d}$ if a downjump occurs. Since an investor is willing to pay $\psi_{0,1;u}$ at time 0 for a payoff vector $[1 \quad 0]^T$ and pay $\psi_{0,1;d}$ at time 0 for a payoff vector $[0 \quad 1]^T$, then:

(13) $$S_{0,1;u}\psi_{0,1;u} + S_{0,1;d}\psi_{0,1;d} = [S_{0,1;u} \quad S_{0,1;d}]\begin{bmatrix}\psi_{0,1;u}\\ \psi_{0,1;d}\end{bmatrix} = S_0.$$

Also, if an investor has the riskless portfolio $[1 \quad 1]^T$, then she is guaranteed to be paid 1 at time 1. As this is equivalent to holding the bond that pays 1 at time 1, then the time 0 price of the portfolio $[1 \quad 1]^T$ is:

(14) $$\psi_{0,1;u} + \psi_{0,1;d} = [1 \quad 1]\begin{bmatrix}\psi_{0,1;u}\\ \psi_{0,1;d}\end{bmatrix} = B_{0,1}.$$

Combining these two vector equations into one matrix equation yields:

(15) $$\begin{bmatrix}S_{0,1;u} & S_{0,1;d}\\ 1 & 1\end{bmatrix}\begin{bmatrix}\psi_{0,1;u}\\ \psi_{0,1;d}\end{bmatrix} = \begin{bmatrix}S_0\\ B_{0,1}\end{bmatrix}.$$

Using inverse matrices, this system can be solved for $\psi_{0,1;u}$ and $\psi_{0,1;d}$.

C. Brownian Motion and Itô Processes

The Central Limit Theorem implies that characteristics of the binomial distribution will approach those of a normal distribution as the number of Bernoulli trials approach infinity. Here, we discuss the approach of a binomial stochastic process to a continuous normal process as the sizes of intervals become smaller and their number approaches infinity. Brownian motion is derived as a limit of a discrete random walk in the appendix to this chapter. We will focus on continuous time-space models, beginning with Brownian motion, which is a Markov and martingale process. A continuous time-space Markov process is also known as a *diffusion process*. Brownian motion processes are probably the simplest of diffusion processes. We will introduce the processes and certain features of them here and perform various operations on them in the next chapter.

Brownian Motion Processes

One particular version of a continuous time/space random walk is a standard *Brownian motion process* Z_t, also called a Wiener process. A process Z_t is a standard Brownian motion process if:

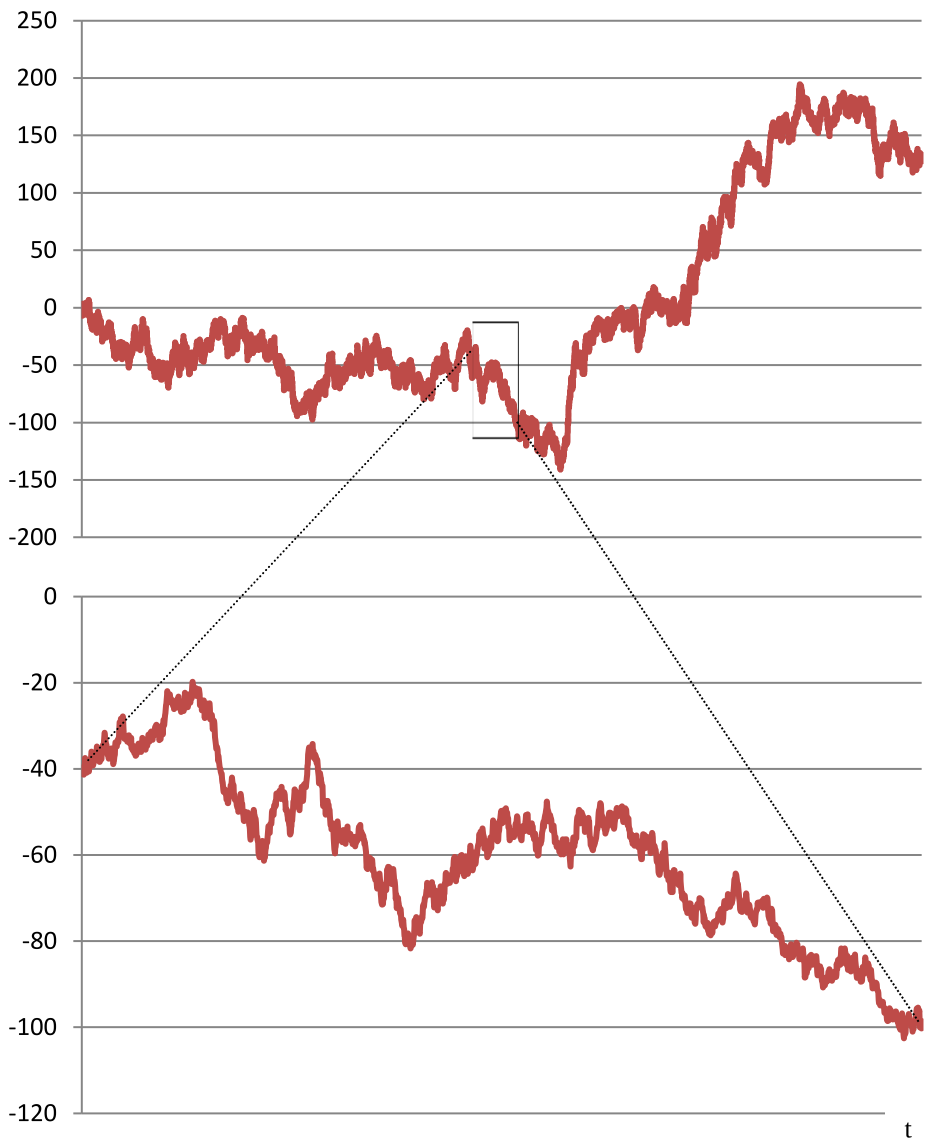

Figure 3: Brownian Motion: A Fractal

1. Changes in Z_t over time are independent over disjoint intervals of time; that is, COV(Z_s-Z_τ, Z_u-Z_v,) = 0 when $s > \tau > u > v$.
2. Changes in Z_t are normally distributed with $E[Z_s-Z_\tau] = 0$ and $E[(Z_s-Z_\tau)^2] = s - \tau$ for $s > \tau$. Thus, $Z_s - Z_\tau \sim N(0, s-\tau)$ with $s > \tau$.
3. Z_t is a continuous function of t.
4. The process begins at zero, $Z_0 = 0$.

Standard Brownian motion is a martingale, since for $s < t$, the process satisfies the two conditions for our martingale definition set forth earlier. The first is:

$$E[Z_t|\mathcal{F}_s] = E[(Z_t - Z_s) + Z_s|\mathcal{F}_s] = E[(Z_t - Z_s)|\mathcal{F}_s] + E[Z_s|\mathcal{F}_s] = Z_s.$$

Since $Z_t \sim N(0, t)$, the following verifies our second martingale condition from Section A:

$$E[|Z_t|] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} |z| e^{-\frac{1}{2t}z^2}\, dz = \frac{2}{\sqrt{2\pi t}} \int_{0}^{\infty} z e^{-\frac{1}{2t}z^2}\, dz = \frac{-2\sqrt{2t}}{\sqrt{\pi}} e^{-\frac{1}{2t}z^2}\Big|_0^{\infty} = \frac{2\sqrt{2t}}{\sqrt{\pi}} < \infty.$$

A random walk consists of taking random discrete unit steps at discrete times $t=0,1,2,\ldots$ along a one-dimensional number line (axis). Brownian motion consists of continuous random movement along a one-dimensional number line over continuous time $t \geq 0$. If we let z denote the possible values on the z-axis for Brownian motion Z_t, then as time progresses (increases) the values of Z_t will move up and down the z-axis in a random but continuous way. Just as a random walk results in a particular path for each specific history in its sample space, a specific history ω in the sample space Ω for Brownian motion results in a particular Brownian motion path $Z_t(\omega)$. One can then graph Brownian motion $Z_t(\omega)$ versus time t for a specific history ω. Figure 7.3 depicts a plot of $Z_t(\omega)$ versus time t for a particular specific history ω in its sample space along with a close-up of a short segment of the process (note the changes in axis scaling).

Brownian motion has a number of interesting traits. First, it is continuous everywhere and differentiable nowhere (it never smooths) under Newtonian calculus; the Brownian motion process is not smooth and does not become smooth as time intervals decrease. This is because along any interval of time Δt, the change in height of the Brownian motion, $Z_{t+\Delta t} - Z_t$, is on the order of its standard deviation, which equals $\sqrt{\Delta t}$. Thus, $(Z_{t+\Delta t} - Z_t)/\Delta t$ is on the order of $1/\sqrt{\Delta t}$, which approaches infinity as $\Delta t \to 0$. This means that the average rate of change of Brownian motion over any time interval approaches infinity as the width of the time interval approaches zero. We see in Figure 3 that Brownian motion is a *fractal*, meaning that regardless of the length of the observation time period, the process can be defined equivalently by a simple change in the time scale. Graphically, this results in Brownian motion appearing similar no matter what scale of time one studies the graph if one is careful to scale the horizontal time scale quadratically faster than the vertical scale. This is because the horizontal scale is $\Delta t = (\sqrt{\Delta t})^2$, which is the square of the vertical scale. Consider the Brownian motion process represented by the top graph in Figure 3. If a short segment of is cut out and magnified as in the bottom graph in Figure 3, the graphs look very similar to one another in that they don't smooth and the variance is proportional to time. Further magnifications of cutouts would continue to result in the same phenomenon. In addition, once a Brownian motion hits a given value, it will return to that value infinitely often. More generally, we will scale the variance by a multiple σ^2 so that the change in the process S_t from time t to time $t+\Delta t$ takes the form:

$$\Delta S_t = \sigma \Delta Z_t = \sigma(Z_{t+\Delta t} - Z_t) \sim N(0, \sigma^2 \Delta t)\,.$$

Consider the (arithmetic) Brownian motion process S_t of the form: $S_t = S_0 + \sigma Z_t$ where S_0 and $\sigma>0$ are constants, and Z_t is standard Brownian motion. The graph of a particular possible path of S_t versus time t is similar to the graph of standard Brownian motion. It is continuous but nondifferentiable (zig-zag shape). The difference is that it can start (time 0) at $S_0 \neq 0$ rather than at the origin 0. Furthermore its variance may be different (when $\sigma^2 \neq 1$) from the variance for standard Brownian motion. S_t has a normal distribution with mean S_0 and variance $\sigma^2 t$ ($S_t \sim$

$N(S_0, \sigma^2 t))$ since:

$$\frac{S_t - S_0}{\sigma\sqrt{t}} = \frac{Z_t}{\sqrt{t}} = Z$$

where $Z \sim N(0,1)$. Note that $\Delta S_t = S_{t+\Delta t} - S_t \sim N(0,\sigma^2 \Delta t)$.This Brownian motion process is said to have a unit variance of σ^2 since when $\Delta t=1$ the variance of $S_{t+1} - S_t$ equals σ^2.

Illustration: Brownian Motion

Suppose a stock price S_t follows a Brownian motion process with an initial stock value of \$50 and a unit variance of $\sigma^2 = 4$. Suppose that we want to find the probability that the stock price is less than \$56 at time 3. Since $S_t \sim N(50,4t)$, then $S_3 \sim N(50,12)$. Using a standard z-Table, we find that this probability equals:

$$P(S_3 < 56) = P\left(\frac{S_3 - 50}{\sqrt{12}} < \frac{56 - 50}{\sqrt{12}}\right) = P(Z < 1.73) = .9582.$$

If one were to graph this stock price as a function of time, the shape might look similar to Figure 1 above until the present time. After the present time, we would not yet know the shape of the graph since the values of the stock are unknown. What we can say is that the graph will be one of an infinite number of possible choices of Brownian motion paths of the type in Figure 1. At any given moment of time t in either the present or future, the stock price S_t will change by unknown random amount $\Delta S_t = S_{t+\Delta t} - S_t$ either up or down over the time interval $[t, t+\Delta t]$ so that this change is normally distributed with variance $4t$.

<u>Brownian Motion Processes with Drift</u>

In Chapter 3, we studied and solved a number of financial models based on based on deterministic differential equations and obtained their solutions. However, since the future is unknown, more realistic differential models will require terms that are probabilistic and random in nature. Just as $dx(t) = x(t+dt)-x(t)$ denotes the infinitesimal change of a real valued function $x(t)$ resulting from an infinitesimal change dt in time, $dZ_t = Z_{t+dt} - Z_t$ will denote an infinitesimal change in Brownian motion Z_t resulting from the time change dt. For each fixed t and dt, by property (2) of Brownian motion, $dZ_t = Z_{t+dt} - Z_t \sim N(0,dt)$ is a random variable having a normal distribution with mean 0 and variance dt.

In order to create probabilistic models for a security S_t, we will generalize further on the Brownian motion process to allow for drift. For now, we provide a basic introduction here where we will now allow for drift a in the process as follows:

$$dS_t = a\,dt + b\,dZ_t$$

where a represents the drift tendency in the value of S_t, dZ_t is the infinitesimal change in the standard Brownian motion process and b is a scaling factor for standard deviation in this process. In a sense, b can represent the instantaneous standard deviation of returns for a stock whose returns follow this Wiener process. If a and b are constants, then this process is called arithmetic Brownian motion with drift. Because prices of many securities such as stocks tend to have a predictable drift component in addition to randomness, generalized Wiener processes might be more practical for modeling purposes than standard Brownian motion, which only includes a random element.[8] The generalized Wiener process expression can be applied to stock returns as follows:

(25) $$dS_t / S_t = \mu dt + \sigma dZ_t$$

The drift term, μ, represents the instantaneous expected rate of return for the stock per unit of time and σ is the instantaneous stock return standard deviation. A process S_t that follows equation (25) is called a geometric Wiener process (geometric Brownian motion), and it is the primary process that is used in finance to model stock prices S_t. We will derive the Black-Scholes Option Pricing Model from this Wiener process shortly.

This geometric Wiener process can be interpreted in a stock environment as a return generating process. Again, the Brownian path σdZ_t of this process is not Newtonian differentiable. This means that the path does not smooth, so that we cannot draw tangent lines that we would otherwise associate with first derivatives.

5.3.5 Itô Processes

An *Itô process*, defined as a function of one or more stochastic variables such as S_t and one or more deterministic variables such as t can be characterized similarly to the following function of S_t and t:

(26) $$dS_t = a(S_t, t)dt + b(S_t, t)dZ_t$$

where a and b represent drift and variability terms which may change over time. Note that both the drift and variance terms, a and b, are functions of both S_t and t, and may change over time. We will not attempt to solve stochastic differential equations that take the form of equations (25) or (26) in this chapter. We merely seek to motivate the concept to more smoothly transition to in-depth coverage in Chapters 9 and 10.

D. Option Pricing: A Heuristic Derivation of Black Scholes[9]

Here, we will make our first effort at deriving the Black Scholes options pricing model, reasoning through a rather heuristic derivation, and follow up with more rigorous derivations in Chapters 9 and 10. The derivation of the price for the call option provided in this section is not based on the powerful techniques that will be developed in later chapters, yet it produces the same results. The advantage of this heuristic approach is that the derivation is easier. However, it does not develop the rationale underlying risk-neutral pricing aspects for options. We will make all standard Black-Scholes assumptions, including that that investors price options as though they are risk neutral. In this section, we will derive the value of a call. The expected future value of the call is:

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[8] A Wiener Process is a continuous time-space Markov Process with normally distributed increments. A positive drift would be consistent with time value of money and investor risk aversion.

[9] The Black-Scholes Model will be thoroughly discussed in Chapter 10 and the essential mathematics are introduced here and in Chapter 9. This heuristic derivation is in the spirit of Boness [1964] and Sprenkle [1964] and as described by Jarrow and Rudd [1983].

$$E[c_T] = E[MAX[S_T - X), 0]] = \int_X^{\infty} (S_T - X)p(S_T)dS_T$$

$$= \int_X^{\infty} S_T p(S_T)dS_T - X\int_X^{\infty} p(S_T)dS_T = \int_X^{\infty} S_T p(S_T)dS_T - XP(S_T > X)$$

where $p(S_T)$ is the density function for the random variable S_T.[10]

Estimating Exercise Probability in a Black-Scholes Environment

The value of a stock option is directly related to the probability that it will be exercised. That is, the option value is related to the probability that expiry date stock price S_T exceeds X, the exercise price of the option. The assumption that the stock price follows a geometric Brownian motion process means that the logarithmic return of a stock follows an arithmetic Brownian motion and is normally distributed with an upward drift $\hat{\mu}T$ taking the following form:

(27) $$ln\left(\frac{S_T}{S_0}\right) = \hat{\mu}T + \sigma Z_T$$

with S_T having the following probability distribution:

$$S_T = S_0 e^{\hat{\mu}T + \sigma\sqrt{T}Z}$$

where $Z \sim N(0,1)$.[11] The drift constant $\hat{\mu}$ is called the mean logarithmic stock return rate. In order to obtain the risk-neutral price of a stock option, we will show in chapter 9 that $\hat{\mu} = r - \frac{\sigma^2}{2}$, with r being the riskless return rate. We will assume this result for now.

To price the option, we begin by finding the probability that $S_T > X$:

(28) $$P(S_T > X) = P\left(S_0 e^{\hat{\mu}T + \sigma\sqrt{T}Z} > X\right) = P\left(\hat{\mu}T + \sigma\sqrt{T}Z > \ln\left(\frac{X}{S_0}\right)\right)$$

$$= P\left(Z > \frac{\ln\left(\frac{X}{S_0}\right) - \hat{\mu}T}{\sigma\sqrt{T}}\right) = P\left(Z > -\frac{\ln\left(\frac{S_0}{X}\right) + \hat{\mu}T}{\sigma\sqrt{T}}\right)$$

$$= P\left(Z < \frac{\ln\left(\frac{S_0}{X}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) = N\left(\frac{\ln\left(\frac{S_0}{X}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$$= N(d_2),$$

if we define d_2 to equal $\frac{\ln\left(\frac{S_0}{X}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$. Thus, $S_T > X$ whenever our normally distributed random variable Z is less than $d2$. This means that $N(d_2)$ is the probability that the option will be

[10] The derivation improves if we merely substitute risk neutral densities $q(S_T)$ for physical densities $p(S_T)$.

[11] Remember that our random variable Z_t follows a Brownian motion process, and is, therefore, normally distributed.

exercised. So, look up the value for d_2 on a z-table and you have the probability that the option will be exercised.

The Expected Expiry Date Call Value

Next, we will focus on the term:

$$\int_X^{\infty} S_T p(S_T) dS_T.$$

Since $S_T = S_0 e^{\hat{\mu}T+\sigma\sqrt{T}Z}$ and $S_T > X$ is equivalent to Z > $-d_2$, (which is the same as Z < d_2 since the normal curve is symmetric), it follows that:

$$\int_X^{\infty} S_T p(S_T) dS_T = S_0 \int_{-d_2}^{\infty} e^{\hat{\mu}T+\sigma\sqrt{T}z} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} = S_0 \int_{-d_2}^{\infty} e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}$$

$$S_0 e^{rT} \int_{-d_2}^{\infty} e^{\left(-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} = S_0 e^{rT} \int_{-d_2}^{\infty} e^{-\frac{1}{2}(-\sigma\sqrt{T}+z)^2} \frac{dz}{\sqrt{2\pi}}$$

Next, we rewrite based on an algebraic manipulation involving "completing the square:"[12]

$$\int_X^{\infty} S_T p(S_T) dS_T = S_0 e^{rT} \int_{-d_2}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{T})^2} \frac{dz}{\sqrt{2\pi}}$$

Make the change of variables $y = z - \sigma\sqrt{T}$, which yields:

$$\int_X^{\infty} S_T p(S_T) dS_T = S_0 e^{rT} \int_{-d_2-\sigma\sqrt{T}}^{\infty} e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}$$

$$= S_0 e^{rT} \int_{-\infty}^{d_2+\sigma\sqrt{T}} e^{-\frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}} = S_0 e^{rT} N(d_1)$$

where we define $d_1 = d_2 + \sigma\sqrt{T}$. Thus, the expected expiry date call value equals:

$$E[c_T] = \int_X^{\infty} S_T p(S_T) dS_T - XP(S_T > X) = S_0 e^{rT} N(d_1) - XN(d_2).$$

Now discount this expected future value to obtain the Black-Scholes Options Pricing Model:

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[12] If we focus on the exponents of the equations above and below, with a little algebra, we can verify that $\hat{\mu}T + \sigma\sqrt{T}z - \frac{z^2}{2} = \left(\hat{\mu} + \frac{\sigma^2}{2}\right)T - \frac{(z-\sigma\sqrt{T})^2}{2} = rT - \frac{(z-\sigma\sqrt{T})^2}{2}$. See Exercise 10 at the end of the chapter.

(29) $$c_0 = S_0 N(d_1) - \frac{X}{e^{rT}} N(d_2)$$

Observations Concerning $N(d_1)$, $N(d_2)$ and c_0

With some minor manipulation, the Black-Scholes options pricing model provides several useful interpretations concerning call value:

1. The probability that the stock price at time T will exceed the exercise price X of the call is $P[S_T > X] = N(d_2)$.
2. The expected value of the stock conditional on the stock's price S_T exceeding the exercise price of the call is:

$$E[S_T | S_T > X] = \frac{\int_X^\infty S_T p(S_T) dS_T}{\int_X^\infty p(S_T) dS_T} = \frac{S_0 e^{rT} N(d_1)}{N(d_2)}$$

3. The expected expiry date call value is simply the product of the probability of call exercise and the expected value of the stock conditional on the stock's price exceeding the exercise price of the call, minus the expected exercise value paid at time T:

$$E[c_T] = E[S_T | S_T > X] \, P[S_T > X] - XP(S_T > X)$$
$$= \frac{S_0 e^{rT} N(d_1)}{N(d_2)} N(d_2) - XN(d_2) = S_0 e^{rT} N(d_1) - XN(d_2).$$

4. The present value of the call is simply the discounted value of its expected future value:

$$c_0 = E[c_T] e^{-rT} = e^{-rT}[S_0 e^{rT} N(d_1) - XN(d_2)] = S_0 N(d_1) - Xe^{-rT} N(d_2).$$

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Exercises

1. Suppose that a particular swap contract is currently valued at 0 and that the probability equals *p* that the price of the contract will increase by 1 at times 1 and 2. The probability that the price of the contract will decrease by 1 equals 1-*p* at times 1 and 2. Assume that the event that the price increases or decreases at time 1 is independent of the event that the price increases or decreases at time 2. No numerical calculations are needed for this example.

   a. What are all of the possible distinct price 2-period change outcomes for the contract through time 2? For example, let the notation (1,-1) mean that the contract's price will have increased by 1 at time 1 and decreased by 1 at time 2.
   b. The sample space for this process is Ω = {(1,1), (1,-1), (-1,1), (-1,-1)}.
   c. What are the potential time zero events? What are the probabilities associated with each of the potential time zero events?
   d. What is $P(X_1 = 1)$?
   e. What is $P(\{(1,1), (1,-1)\})$?
   f. Is the following statement true: $P(X_1 = -1) = P(\{(-1,1), (-1,-1)\}) = 1 - p$?
   g. Is the following statement true: $P(X_1 = 1\ or\ X_1 = -1) = P(\emptyset) = 0$?
   h. What is $P(X_2 = 0\ or\ X_2 = 2) = P(\{(1,1), (1,-1), (-1,1)\}) = 2p(1-p) + p^2 = p(2-p)$.
   i. What is $P(X_1 = 1\ and\ X_2 = 0)$?

2. Suppose that a brokerage firm uses an algorithm so that the number of portfolios that it assigns to any given broker satisfies the following model. Let X_t denote the number of portfolios that it assigns a broker on day *t.* Assume that the firm will assign either 1, 2, or 3 portfolios per day to a broker. At time $t = 1$ the random variable X_1 has probabilities equal to1/3 of taking on the values of 1, 2 or 3. For all subsequent times $t = 2, 3, 4, \ldots$, the variable X_t satisfies the following conditions. If X_{t-1}=1, then $P(X_t = 2) = ½$ and $P(X_t = 3) = ½$. If $X_{t-1} = 2$, then $P(X_t = 1) = ½$ and $P(X_t = 3) = ½$. If X_{t-1}=3, then $P(X_t = 1) = ½$ and $P(X_t = 2) = ½$.

   a. Is the process X_t stochastic?
   b. Is the process X_t a Markov process?
   c. Are the increments $Z_t = X_t - X_{t-1}$ independent over time starting with $t = 2$?

3. Cards are dealt one at a time from a standard 52-card randomly shuffled deck and points are awarded to the lone recipient based on the number on the card (2 to 10) or 11 if the dealt card is a "face card" or Ace. Let S_t represent the number of points to be held by the recipient after *t* cards have been dealt by the dealer. For parts a through c, suppose that the cards have been dealt without replacement. For parts d,e, f and g, assume that the cards have been dealt with replacement and that 1 point is awarded if the number on the card is a 2 through a 6, 0 points are awarded if the number on the card is 7, 8, or 9, and -1 point is awarded if the card is a ten, a face card, or an ace. We point out that this is the most common point systems used by card counters playing blackjack. Note that we are assuming a finite process since the "time" remains finite, running from 0 to 52.

   a. Is this process stochastic?
   b. Is this process Markov?
   c. Is this process a submartingale?
   d. Is this process Markov?

e. Is this process a submartingale?
f. Is this process Markov?
g. Is this process a martingale?

4.a. Consider two probabilities $\mathbb{P}$ and $\mathbb{Q}$ defined on the sample space $\Omega=\{\omega_1,\omega_2, \omega_3\}$ in the following way: $p(\omega_1)=0$, $p(\omega_2)=.4$, $p(\omega_3)=.6$, and $q(\omega_1)=.4$, $q(\omega_2)=0$, $q(\omega_3)=.6$. Are $\mathbb{P}$ and $\mathbb{Q}$ equivalent probability measures?
b. Now, we change our exercise prices. Suppose that, instead, $p(\omega_1)=0$, $p(\omega_2)=.4$, $p(\omega_3)=.6$, and $q(\omega_1)=0$, $q(\omega_2)=.3$, $q(\omega_3)=.7$. Are $\mathbb{P}$ and $\mathbb{Q}$ equivalent probability measures now?

5. Suppose that a particular stock will experience $n = 4$ consecutive transactions over a given period and that we wish to know the probability that the stock price will have increased in exactly y^* of those four transactions. Assume that the transactions follow a binomial process.
a. What is the number of potential orderings of price increases (+) and decreases (-) in these stock prices over the 4 transactions?
b. List all potential orderings of the directional changes of the stock price (+) or (-).
c. Suppose that each transaction is equally likely to result in a price increase or decrease, then each trial has an equal probability of an upjump or downjump ($p = ½$). What is the probability that any one of the potential orderings listed in part c will be realized?
d. Suppose that the probability of a price increase in any given transaction equals $p = .6$. What is the probability of realizing three price increases followed by a single decrease?
e. What is the probability that exactly $y^*=3$ price increases will result from $n = 4$ transactions where $p = .6$?
f. What is the probability that more than 3 price increases will result from $n = 4$ transactions where $p = .6$?

6. Examination of trade-by-trade data for a given stock reveals that the stock has a 51% probability of increasing by $.0625 on any given transaction and a 49% probability of decreasing by $.0625 on any given transaction. The stock has a current market value equal to $100 and is expected to trade ten times per day starting today.
a. What is the probability that the stock's price will exceed $99.99 at the end of today?
b. What is the probability that the stock's price will exceed $100.49 at the end of today?
c. What is the probability that the stock's price will exceed $110 at the end of 10 days?

7. Consider a Markov process S_t that produces one of two potential outcomes at each time $t = n\Delta t$ for $n = 0, 1, 2, ...$ For example, suppose a stock price can increase (uptick) by $a\sqrt{\Delta t}$ $(a>0)$ with physical probability p or decrease (downtick) by $a\sqrt{\Delta t}$ with physical probability$(1-p)$. This process applies to each time period $t \geq \Delta t$. Assume that each uptick or downtick at a particular time is independent of the uptick or downtick at any other time, and so S_t follows a binomial process.
a. Write a function that provides the expected value of $S_{t+\Delta t}$ at time $t + \Delta t$ given S_t.
b. Under what circumstances is this Markov Process also a Martingale?
c. Derive a formula to obtain the variance of the process $S_{t+\Delta t} - S_t$.
d. Simply your formula in parts a and c for p = .5.
e. What is the standard deviation of this process?

8. Kestrel Company stock is currently selling for \$40 per share. Its historical standard deviation of returns is .5; this historical standard deviation will be used as a forecast for its standard deviation of returns. Assume that the logarithmic return on the stock follows a Brownian motion with drift process. The one-year Treasury Bill rate is currently 5%. What is the probability that the value of the stock will be less than \$30 in one year?

9. Verify that the following two expressions to define d_2 are identical. That is, verify that $d_2 = d_1 - \sigma\sqrt{T}$:

$$\frac{\ln\left(\frac{S_0}{X}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{S_0}{X}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} - \sigma\sqrt{T}$$

This exercise requires only basic algebra, and is useful because both versions of this d_2 expression appear in the literature and some more advanced derivations are more convenient when starting with one rather than the other.

10. Verify Footnote 12 for Section D, which states the following:

$$\hat{\mu}T + \sigma\sqrt{T}z - \frac{z^2}{2} = \left(\hat{\mu} + \frac{\sigma^2}{2}\right)T - \frac{(z - \sigma\sqrt{T})^2}{2}.$$

11. Let $P[S_T > X] = \int_X^\infty p(S_T)dS_T$ equal the probability that a given call will be exercised at expiry, time T, where S_T is the expiry date random stock price and X is the exercise price of the option. Do not assume that the Black-Scholes assumptions are necessarily true (avoid use of $N(d_1)$ and $N(d_2)$).

a. Write a function that gives the expected expiry date value of the stock conditional on its value exceeding X.

b. Write a function that gives the expected expiry date value of the call conditional on its underlying stock value exceeding X.

c. Based on your answers to parts a and b, write a function that gives the expected expiry date value of the call.

12. A *gap option* has two exercise prices, one (X) that triggers the option exercise and the second (M) that represents the price or cash flow that exchanges hands at exercise. For example, a gap call with exercise prices $X = 50$ and $M = 40$ enables its owner to purchase its underlying stock for 40 when its price rises above 50. The European version of this option can be exercised only when the underlying stock price exceeds 50 at expiration, but at a price of $M = 40$. Assume that the underlying stock price follows the following process:

$$S_T = S_0 e^{\left[\mu T + \sigma\sqrt{T}z\right]}$$

The current stock price is $S_0 = 45$, the variance of underlying stock returns equals $\sigma^2 = .16$, the riskless return rate equals $r = .05$.

a. What is the probability that the underlying stock will be worth more than 50 in one year?
b. What is the expected value of this stock contingent on its price exceeding 50?
c. The owner of the gap option has the right to pay 40 for the underlying stock if its price rises above 50. What is the expected value of this call contingent on its exercise?
d. What is the current value of this gap call?

Solutions

1.a. All of the possible distinct price 2-period change outcomes for the stock to time 2 are (1,1), (1.-1), (-1,1), and (-1,-1).

b. The sample space for this process is $\Omega = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$.

c. The time zero events are Ω and $\emptyset$. $P(X_0 = 0) = P(\Omega) = 1$ and $P(X_0 \neq 0) = P(\emptyset) = 0$.

d. p

e. $P(\{(1,1),(1,-1)\}) = p$

f. Yes

g. No. $P(X_1 = 1 \; or \; X_1 = -1) = P(\Omega) = 1$

h. $P(X_2 = 0 \; or \; X_2 = 2) = P(\{(1,1),(1,-1),(-1,1)\}) = 2p(1-p) + p^2 = p(2-p)$.

i. Since the σ-algebra $\mathcal{F}_1$ is defined such that $\mathcal{F}_1 \subset \mathcal{F}_2$, this allows us to compute probabilities that involve random variables at different times such as: $P(X_1 = 1 \; and \; X_2 = 0) = P(\{(1,1),(1,-1)\} \cap \{(1,-1),(-1,1)\}) = P(\{(1,-1)\}) = p(1-p)$.

2.a. This is a stochastic process.

b. Clearly this stochastic process is a discrete Markov Process, since the probability that X_t will take a particular value depends only on its current value X_{t-1}.

c. No. First, we will show that $P(X_t = i) = 1/3$ for $i = 1, 2, 3$, and $t = 2, 3, \ldots$. To find $P(X_2 = 1)$, we calculate that

$$P(X_2 = 1) = P(X_2 = 1|X_1 = 1)P(X_1 = 1) + P(X_2 = 1|X_1 = 2)P(X_1 = 2) + P(X_2 = 1|X_1 = 3)P(X_1 = 3) = 0 \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{3}.$$

One can derive in a similar manner that $P(X_2 = 2) = 1/3$ and $P(X_2 = 3) = 1/3$. Using the same type of calculations as above, it follows by mathematical induction that $P(X_t = i) = 1/3$ for $i = 1, 2, 3$, and any $t \geq 2$. Suppose that $Z_t = X_t - X_{t-1}$ for $t = 2, 3, 4, \ldots$. For all $t > 1$, $P(Z_t = 1) = 1/3 \times ½ + 1/3 \times ½ + 1/3 \times 0 = 1/3$. Note that this is an unconditional probability. Next, we find $P(Z_{t-1}=1, Z_t=1)$. If $Z_t=1$, then either $X_{t-1} = 1$ or $X_{t-1} = 2$. But if we also have $Z_{t-1}=1$, then we must have $X_{t-2} = 1$. Thus, $P(Z_{t-1}=1,Z_t=1)=P(X_{t-2}=1,X_{t-1}=2,X_t=3)=P(X_t=3|X_{t-2}=1,X_{t-1}=1)P(X_{t-2}=1,X_{t-1}=1)= P(X_t=3|X_{t-2}=1,X_{t-1}=1)P(X_{t-1}=2|X_{t-2}=1)P(X_{t-2}=1)=1/2\times 1/2\times 1/3=1/12$. This means that $P(Z_{t-1}=1,Z_t=1)=1/12\neq 1/9=1/3\times 1/3= P(Z_{t-1}=1)P(Z_t=1)$. This proves that the increments are not independent.

3.a. Yes: For every fixed number of cards t that have been dealt, the sum S_t of the values of the t dealt cards is a random variable. Furthermore, this random variable is indexed by time (technically, by the number of cards that have been dealt). The common probability space is the probability space that arises from the set of outcomes of all possible ways to deal 52 cards from the deck.

b. No: It is enough to show the following one case: $P(S_3=9|S_0=0,S_1=2,S_2=6)\neq P(S_3=9|S_2=6)$ to justify that the Markov property is violated. Note that $Z_t = S_t - S_{t-1}$ denotes the value of the t-th card that is dealt. Thus, $P(S_3=9|S_0=0,S_1=2,S_2=6)= P(Z_3=3|Z_1=2,Z_2=4)$ and $P(S_3=9|S_2=6)= P(Z_3=3|Z_1+Z_2=6)$. We calculate that:

$$P(Z_3 = 3|Z_1 = 2, Z_2 = 4) = \frac{P(Z_1=2,Z_2=4,Z_3=4)}{P(Z_1=2,Z_2=4)} = \frac{\frac{4\times4\times4}{52\times51\times50}}{\frac{4\times4}{52\times51}} = \frac{2}{25}.$$

In order that Z_1+Z_2 =6, either $Z_1=2$ and $Z_2 =4$, or $Z_1=4$ and $Z_2 =2$, or $Z_1=3$ and $Z_2 =3$. Thus:

$$P(Z_3 = 3|Z_1 + Z_2 = 6) = \frac{P(Z_1 + Z_2 = 6, Z_3 = 3)}{P(Z_1 + Z_2 = 6)}$$

$$= \frac{P(Z_1 = 2, Z_2 = 4, Z_3 = 3) + P(Z_1 = 4, Z_2 = 2, Z_3 = 3) + P(Z_1 = 3, Z_2 = 3, Z_3 = 3)}{P(Z_1 = 2, Z_2 = 4) + P(Z_1 = 4, Z_2 = 2) + P(Z_1 = 3, Z_2 = 3)}$$

$$= \frac{\frac{4\times4\times4}{52\times51\times50} + \frac{4\times4\times4}{52\times51\times50} + \frac{4\times3\times2}{52\times51\times50}}{\frac{4\times4}{52\times51} + \frac{4\times4}{52\times51} + \frac{4\times3}{52\times51}} = \frac{19}{275} \neq \frac{2}{25}.$$

c. Yes: Since $E[S_t|S_{t-1}]=S_{t-1}+E[S_t-S_{t-1}|S_{t-1}]$ and $S_t-S_{t-1}>0$, then $E[S_t|S_{t-1}]>S_{t-1}$.
d. Yes: Since the cards are dealt with replacement, then each card is dealt from the originally randomly shuffled deck. This implies that the choice of each card dealt is independent of the choice of the any other card that is dealt. Thus, the random variables $\{Z_t\}$ are independent of one another. As we proved in section 5.1.2, the resulting process is Markov.
e. Yes: By the same reason as in part c.
f. Yes: By the same reason as in part d.
g. Yes: First, we write $E[S_t|S_{t-1}]=S_{t-1}+E[S_t-S_{t-1}|S_{t-1}]$. Note that since $Z_t =S_t - S_{t-1}$, $S_0=0$, and $S_{t-1}=Z_1+\ldots+Z_{t-1}$, then the random variable $S_t - S_{t-1}$ is independent of the random variable S_{t-1}. This implies that $E[S_t-S_{t-1}|S_{t-1}]= E[S_t-S_{t-1}]=E[Z_t]$. As the t-th card is being dealt, the probability is 1/52 of it being any given card in the deck. There are 4×5=20 cards that have a point value of 1. There are 4×3=12 cards that have a point value of 0. There are 4×5=20 cards that have a point value of -1. Thus:

$$E[Z_t] = 1 \times \frac{20}{52} + 0 \times \frac{12}{52} + (-1) \times \frac{20}{52} = 0.$$

This means that $E[S_t|S_{t-1}]=S_{t-1}+E[S_t-S_{t-1}|S_{t-1}]=S_{t-1}$. We conclude that S_t is a martingale.

4.a. No: $p(\omega_1)=0$ while $q(\omega_1)=.4\neq0$.
b. Yes: $p(\omega_i)\neq0$ if and only if $q(\omega_i)\neq0$.

5.5.a. $2^4 = 16$
b.

+ + + +	- + + +	- + + -	- + - -
+ + + -	+ + - -	- + - +	- - + -
+ + - +	+ - + -	- - + +	- - - +
+ - ++	+ - - +	+ - - -	- - - -

c. Since each transaction is equally likely to result in a price increase or decrease, then each ordering has equal probability (p = ½). The probability that any particular ordering will be realized is $P[ordering] = p^n$=1/2^4 = .0625.
d. The probability of three price increases followed by a single decrease (+++-) equals .0864:

$$P[+,+,+,-] = P[ordering] = p^{y*}(1-p)^{n-y*} = p^3(1-p)^{4-3} = .0864$$

e. The probability that exactly $y^*=3$ price increases will result from $n = 4$ transactions where $p = .6$ is calculated from the following:

$$P(y*) = \binom{n}{y*} p^{y*}(1-p)^{n-y*}$$

$$P(3) = \binom{4}{3}.6^3(1-.6)^{4-3} = \frac{4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 1}\cdot .216\cdot .4 = 4\cdot .0864 = .3456$$

f. $P[y*>3] = P[y*=4] = .6^4 = .1296.$

6.a. At least 5 increases are necessary for the price at the end of the day to exceed $99.99. The probability of this occurring is computed as follows:

$$P[X > 99.99] = \binom{10}{5}.51^5.49^5 + \binom{10}{6}.51^6.49^4 + \binom{10}{7}.51^7.49^3 + \binom{10}{8}.51^8.49^2 + \binom{10}{9}.51^9.49^1 + \binom{10}{10}.51^{10}.49^0$$
$$= 0.2456 + 0.2130 + 0.1267 + 0.0494 + 0.0114 + 0.0012$$
$$= .6474$$

b. At least 9 increases (.50÷.0625 = 8, plus one more gain to offset one loss out of 10) are necessary for the price at the end of the day to be at least $100.50. The probability of this occurring is computed as follows:

$$P[X > 100.49] = \binom{10}{9}.51^9.49^1 + \binom{10}{10}.51^{10}.49^0 = 0.0114 + 0.0012$$
$$= .0126$$

c. zero: the stock would require 160 more upjumps than downjumps in price to increase by 10 points, yet it only has 100 opportunities to upjump.

7.a. Under physical probability measure $\mathbb{P}$, the expected value of $S_{t+\Delta t}$ at time $t + \Delta t$ given S_t is
$E_{\mathbb{P}}[S_{t+\Delta t}|S_t] = p\left(S_t + a\sqrt{\Delta t}\right) + (1-p)\left(S_t - a\sqrt{\Delta t}\right) = S_t + (2p-1)\left(a\sqrt{\Delta t}\right)$

b. This Markov Process is also a Martingale in the case where $p = 1/2$.

c. Since the variances of $S_{t+\Delta t}$ and $S_{t+\Delta t} - S_t$ are equal if S_t is given, the variance of $S_{t+\Delta t}$ at time $t + \Delta t$ given S_t is:

$$Var_{\mathbb{P}}[S_{t+\Delta t} - S_t|S_t] = E_{\mathbb{P}}[(S_{t+\Delta t} - S_t)^2] - (E_{\mathbb{P}}[S_{t+\Delta t} - S_t])^2$$
$$= p\left(a\sqrt{\Delta t}\right)^2 + (1-p)\left(-a\sqrt{\Delta t}\right)^2 - \left[p\left(a\sqrt{\Delta t}\right) + (1-p)\left(-a\sqrt{\Delta t}\right)\right]^2$$
$$= p\left(a\sqrt{\Delta t}\right)^2 + (1-p)\left(-a\sqrt{\Delta t}\right)^2 - p^2(a^2\Delta t) + (1-p)^2(a^2\Delta t) + 2p(1-p)a^2\Delta t$$
$$= 2pa^2\Delta t - 4p^2a^2\Delta t - p^2(a^2\Delta t) + a^2\Delta t + p^2a^2\Delta t - 2p^2a^2\Delta t + 2pa^2\Delta t - 2p^2a^2\Delta t$$
$$= 2pa^2\Delta t - 4p^2a^2\Delta t + 2pa^2\Delta t = 4pa^2\Delta t - 4p^2a^2\Delta t$$
$$= 4p(1-p)a^2\Delta t.$$

d. If $p = .5$, the expected value and variance of $S_{t+\Delta t}$ given S_t are S_t and $a^2\Delta t$, respectively.

e. The standard deviation of $S_{t+\Delta t}$ given S_t equals $a\sqrt{\Delta t}$.

8. We are going to make use of equation (28) with $S_0 = 40$, $X = 30$, $\sigma = .5$, $r = .05$, and $T=1$. We first need to find:

$$d_2 = \frac{ln\left(\frac{40}{30}\right) + (.05 - \frac{1}{2}.5^2)\times 1}{.5\times\sqrt{1}} = .4254.$$

Using equation (28) we obtain:

$$P(S_T < 30) = 1 - P(S_T \geq 30) = 1 - N(.4254) = 1 - .6647 = .3353.$$

9. We will start by simply dropping identical terms on both sides to focus on the differences between these two expressions:

$$\frac{(r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{(r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} - \sigma\sqrt{T}$$

We drop *r* from both sides and finish by noting that:

$$-\frac{\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \frac{\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} - \sigma\sqrt{T} \text{ because } -\frac{\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \frac{\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} - \frac{\sigma^2 T}{\sigma\sqrt{T}}$$

10. Minor simplification and expanding the square yields:

$$\hat{\mu}T + \sigma\sqrt{T}z - \frac{z^2}{2} = \left(\hat{\mu} + \frac{\sigma^2}{2}\right)T - \frac{(z - \sigma\sqrt{T})^2}{2}$$

$$\hat{\mu}T + \sigma\sqrt{T}z - \frac{z^2}{2} = \hat{\mu}T + \frac{\sigma^2}{2}T - \frac{z^2}{2} + 2\frac{\sigma\sqrt{T}z}{2} - \frac{\sigma^2}{2}T$$

$$\sigma\sqrt{T}z = \frac{\sigma^2}{2}T + 2\frac{\sigma\sqrt{T}z}{2} - \frac{\sigma^2}{2}T$$

$$\sigma\sqrt{T}z = +\sigma\sqrt{T}z$$

11.a. From Bayes Rule, the expected expiry date value of the stock conditional on its value exceeding X is: $E[S_T|S_T > X] = \frac{\int_X^\infty S_T p[S_T]dS_T}{\int_X^\infty p[S_T]dS_T}$.

b. The expected expiry date value of the call given that it is exercised is given by:

$$E[c_T|S_T > X] = \frac{\int_X^\infty (S_T - X)p(S_T)dS_T}{\int_X^\infty p(S_T)dS_T}$$

c. Thus, expected expiry date value of the call is simply the product of the last equations and the probability that the stock's day T value exceeds X:

$$E[c_T] = \frac{\int_X^\infty (S_T - X)p(S_T)dS_T}{\int_X^\infty p(S_T)dS_T}\int_X^\infty p(S_T)dS_T = \int_X^\infty (S_T - X)p(S_T)dS_T$$

12.a. The probability of option exercise

$$P[S_T > 50] = N(d_2) = N\left[\frac{\ln\left(\frac{S_0}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right] = N\left[\frac{\ln\left(\frac{45}{50}\right) + \left(.05 - \frac{.16}{2}\right)}{.4}\right] = N(-.3384) = .36753$$

b.

$$E\left[S_T \middle| S_T > X\right] = \frac{S_0 e^{rT} N(d_1)}{N(d_2)} = \frac{45 \times e^{.05\times 1} \times N\left[\frac{\ln\left(\frac{45}{50}\right) + \left(.05 + \frac{.16}{2}\right)}{.4}\right]}{.36753}$$

$$= \frac{45 \times e^{.05} \times .52456}{.36753} = 67.52$$

c.

$$E\left[c_{GAP,T}\right] = (E\left[S_T \middle| S_T > X\right] - M)N(d_2) = (67.52 - 40) \times .36753 = 10.114$$

d. $$c_{GAP,0} = \frac{E[c_{GAP,T}]}{e^{rT}} = \frac{10.114}{e^{.05\times 1}} = 9.62$$