Chapter 8: Binomial Option Pricing

A. Binomial Option Pricing: One-Period Case

The Binomial Option Pricing Model is based on the assumption that the underlying stock price is a Bernoulli trial in each period, such that it follows a binomial multiplicative return generating process. This means that for any period following a particular outcome, the stock's value will be only one of two potential constant values. For example, the stock's value at time $t+1$ will be either u (multiplicative upward movement) times its prior value S_t or d (multiplicative downward movement) times its prior value S_t.

Notice that we have not specified probabilities of a stock price increase or decrease during the period prior to option expiration. Nor have we specified a discount rate for the option or made inferences regarding investor risk preferences. We will value this call based on the fact that during this single time period, we can construct a riskless hedge portfolio consisting of a position in a single call and offsetting positions in α shares of stock. This means that by purchasing a single call and by selling α shares of stock, we can create a portfolio whose value is the same regardless of whether the underlying stock price increases or decreases. The ratio of the number of shares to offset each call in the portfolio is called the *hedge ratio*, or in the multi-period framework, the *dynamic hedge*. Let us first define the following terms:

X = Exercise price of the stock
S_0 = Initial stock value
u = Multiplicative upward stock price movement
d = Multiplicative downward stock price movement
c_u = MAX[0,uS_0 - X]; Value of call if stock price increases
c_d = MAX[0,dS_0 - X]; Value of call if stock price decreases
α = Hedge ratio
r = Riskless return rate

<u>The Hedge Ratio</u>

In a one-time binomial framework where there exists a riskless asset, we can hedge the call against the stock such the resultant portfolio with one call and α shares of stock will produce the same cash flow whether the stock increases or decreases:

(1) $$c_u + \alpha u S_0 = c_d + \alpha d S_0 = (c_0 + \alpha S_0)(1+r)$$

This one-time period hedge implies a hedge ratio α, which provides for the number of shares per call option position to maintain the perfect hedge (portfolio that replicates the bond):

(2) $$\alpha = \frac{c_u - c_d}{S_0(d-u)}$$

Negative values for α, the hedge ratio, which will always be the case when the call is purchased, imply that α shares of stock are shorted for each call that is purchased.

Multi-period models lead to 2^T potential outcomes without recombining, or, in many instances, $T+1$ potential outcomes with recombining. Thus, complete capital markets require a set of 2^T or T priced securities (stocks or options) with payoff vectors in the same payoff space such that the set of payoff vectors is independent. In multiple period frameworks, hedging with many securities guarantees hedged portfolios at the portfolio termination or liquidation dates. In a binomial framework, there is an important exception. If the hedge α_t can be updated each period t, we write the hedge ratio for period t as follows:

$$\alpha_t = \frac{c_{u,t} - c_{d,t}}{S_t(d-u)}$$

The dynamic hedge is updated each period. Any portfolio employing and updating this dynamic hedge in each period in a binomial framework will be riskless at each period.

<u>Pricing the Call in the One-Period Case</u>

Since the risk of a call can be hedged with an offsetting position in a call, we can perform a little algebra on Equation (1) to value the call in Equation (3) as follows:

(1) $$c_d + \alpha d S_0 = (c_0 + \alpha S_0)(1+r)$$

(3) $$c_0 = \frac{-\alpha S_0(1+r) + c_d + \alpha d S_0}{(1+r)}$$

Equation (3) is the binomial model for one-period pricing for the call. We can add some insight to this by filling in the hedge ratio from Equation 2 as follows, then rearrange terms to simplify and to separate c_u and c_d as follows in Equation (4):

$$c_0 = \frac{-\frac{c_u - c_d}{S_0(d-u)} S_0(1+r) + c_d + \frac{c_u - c_d}{S_0(d-u)} d S_0}{(1+r)}$$

$$c_0 = \frac{\frac{c_u}{(u-d)}(1+r) - \frac{c_d}{(u-d)}(1+r) + \frac{c_d(u-d)}{(u-d)} - \frac{c_u}{(u-d)} d + \frac{c_d}{(u-d)} d}{(1+r)}$$

(4) $$c_0 = \frac{c_u\left[\frac{(1+r)-d}{(u-d)}\right] + c_d\left[\frac{u-(1+r)}{(u-d)}\right]}{(1+r)}$$

Equation (4) has a very nice feature: c_0 can be expressed as the discounted expected future value of the call, which equals the upjump value of the call multiplied by some function of r, u and d plus the downjump value of the call multiplied by another function of r, u and d. These two functions can be interpreted as risk-neutral probabilities, one for upjump and the second for downjump, which is what we will do in the next subsection.

<u>Risk-Neutral Setting: One-Period Case</u>

Here, we will calculate the risk-neutral probabilities. In earlier chapters, we valued a call as a payoff structure identical to a portfolio comprised of underlying shares and bonds. This is what we did above in Equation (3) with the hedge ratio. We use a similar method here, where the equivalent martingale measure is calculated from bond and stock payoffs, and the call is valued based on those risk-neutral probabilities.[1] Valuing the one-time period call in order to obtain its risk-neutral pricing in a single time period binomial framework is straightforward given the risk-neutral probability measure $\mathbb{Q}$, where q represents the risk-neutral probability (from the equivalent martingale $\mathbb{Q}$) of an upjump:

$$E_{\mathbb{Q}}[c_1] = [c_u q + c_d(1-q)] = c_0(1+r).$$

Now, we have a simple expected value model for estimating the expected future value of the call. Solving for c_0 gives the price:

$$c_0 = \frac{c_u q + c_d(1-q)}{1+r}$$

Now, we have the present value of the call. We value shares of the stock relative to the bond as follows:

$$E_{\mathbb{Q}}[S_1] = uS_0 q + dS_0(1-q) = S_0(1+r),$$

which implies that:

$$q = \frac{1+r-d}{u-d}$$

Part of what makes this model so useful is that we do not need to know investor risk-return preferences, the expected return for the stock or even the physical probabilities for upjumps and downjumps. This last point is worth emphasizing: We do not need to know physical probabilities. Instead, we can calculate risk neutral probabilities for upjumps q and downjumps ($1-q$) based on the equivalent martingale. Now, we see that we can price the call in a one-period environment with either of the following:

$$c_0 = \frac{c_u q + c_d(1-q)}{1+r}$$

Illustration: Binomial Option Pricing - One Period Case

Consider a stock currently selling for 10 and assume for this stock that u equals 1.5 and d equals .5. The stock's value in the forthcoming period will be either 15 (if outcome u is realized) or 5 (if outcome d is realized). Consider a one-period European call trading on this particular stock with an exercise price of 9. If the stock price were to increase to 15, the call would be worth 6 ($c_u = 6$); if the stock price were to decrease to 5, the value of the call would be zero ($c_d = 0$). In addition, recall that the current riskless one-year return rate is .25. Based on this information, we should be able to determine the value of the call.

[1] Recall that in a complete market with physical probability measure $\mathbb{P}$, probability measure $\mathbb{Q}$ is said to be an equivalent martingale measure to $\mathbb{P}$ if both are equivalent probability measures, and every discounted security in the market is a martingale with respect to $\mathbb{Q}$. Recall that a stochastic process whose increments have expected value 0, also known as a "random walk."

In our numerical example offered above, we use the following to determine the value of the call in the binomial framework:

$S_0 = 10$	$u = 1.5$	$d = .5$
$c_u = 6$	$c_d = 0$	$r = .25$
$X = 9$		

The risk-neutral probability of an upjump, the hedge ratio and the time zero value of the call in the one-period framework are calculated as follows:

$$q = \frac{1+r-d}{u-d} = \frac{1+.25-.5}{1.5-.5} = .75$$

$$\alpha = \frac{c_u - c_d}{S_0(d-u)} = \frac{6-0}{10(.5-1.5)} = -.6$$

$$c_0 = \frac{c_u q + c_d(1-q)}{1+r} = \frac{6\times.75 + 0\times(1-.75)}{1+.25} = 3.6$$

Recall that the time zero value of the bond was .8. The time zero value of the call is 3.6/.8 = 4.5 times that of the bond; the time zero expected value of the call at time 1 is also 4.5 times the value of the time one bond value:

$$E_{\mathbb{Q}}[c_1] = [c_u q + c_d(1-q)] = 6\times.75 + 0\times(1-.75) = 4.5$$

Thus, under probability measure $\mathbb{Q}$ with the bond as the numeraire, the call price process is a martingale, just as the stock price process is.

B. Multi-Period Framework

Suppose that we express our outcomes in terms of *u* and *d*, such that the numbers of upjumps and downjumps over time determine stock prices. If we are willing to assume that the probability measure is the same in each of these *T* time periods, by invoking the Binomial Theorem, we see that valuing the call in the multi-period binomial setting is similarly straightforward:[2]

$$c_0 = \frac{\sum_{j=0}^{T} \frac{T!}{j!(T-j)!} q^{\,j}(1-q)^{T-j} MAX[u^j d^{T-j} S_0 - X, 0]}{(1+r)^T}$$

The number of computational steps required to solve this equation is reduced if we eliminate from consideration all of those outcomes where the option's expiration date price is zero. Thus, *a*, the smallest non-negative integer for j where $S_T > X$ is given as follows:[3]

[2] See end-of chapter exercise 3 for a derivation. See also Cox and Rubenstein [1985].

[3] We obtain *a* by first determining the minimum number of price increases *j* needed for S_T to exceed *X*:

$$S_T = u^j d^{T-j} S_0 > X$$

We then solve this inequality for the minimum positive integer value for *j* with $u^j d^{T-j} S_0 > X$ (note: if $j \leq 0$, *a* will equal

(19) $$a = INT\left[MAX\left[\frac{\ln\left(\frac{X}{S_0 d^T}\right)}{\ln\left(\frac{u}{d}\right)}, 0 \right] + 1 \right]$$

We can simplify the Binomial Model further by substituting *a* and rewriting as follows:

(20) $$c_0 = \frac{\sum_{j=a}^{T} \frac{T!}{j!(T-j)!} q^j (1-q)^{T-j} \left[u^j d^{T-j} S_o - X \right]}{(1+r)^T}$$

or:

$$c_0 = S_0 \left[\sum_{j=a}^{T} \frac{T!}{j!(T-j)!} q^j (1-q)^{T-j} \frac{u^j d^{T-j}}{(1+r)^T} \right] - \frac{X}{(1+r)^T} \left[\sum_{j=a}^{T} \frac{T!}{j!(T-j)!} q^j (1-q)^{T-j} \right]$$

or, in short-hand form:[4]

$$c_0 = S_0 B[T, q'] - X(1+r)^{-T} B[T, q]$$

where $q' = qu/(1+r)$ and $1-q' = d(1-q)/(1+r)$. The values q', q and T are the parameters for the two binomial distributions. Three points are worth further discussion regarding this simplified Binomial model:

1. First, as T approaches infinity, the binomial distribution will approach the normal distribution, and the binomial model will approach the Black Scholes model, which we will discuss later in this chapter and in Chapter 6.
2. The current value of the option is:

$$c_0 = \frac{E[c_T]}{(1+r)^T} = P[S_T > X] \frac{E[S_T | S_T > X]}{(1+r)^T} - \frac{X}{(1+r)^T} P[S_T > X]$$

   First, this implies that the binomial distribution $B[T, q] = P[S_T > X]$ provides the probability that the stock price will be sufficiently high at the expiration date of the option to warrant its exercise. Second, $S_0 B[T, q']/B[T, q]$ can be interpreted as the discounted expected future value of the stock conditional on its value exceeding X.
3. The call is replicated by a portfolio comprised of a long position in $B[T,q'] < 1$ shares

0). First divide both sides of the inequality by $S_0 d^T$ so that $(u/d)^j > X/(S_0 d^T)$. Next, take logs of both sides to obtain: $j \ln(u/d) > \ln(X/(S_0 d^T))$. Finally, divide both sides by $ln(u/d)$ to get the desired result. Thus, a is the smallest positive integer for j such that $S_T > X$.

[4]As the lengths of time periods approach zero, d must approach $1/u$.

of stock and borrowings. Investment in stock totals $S_0B[T,q']$ and borrowings total $X(1+r)^{-T}B[T, q]$. The replication amounts must be updated at each time period.

Extending the Binomial Model to Two Periods

Now, we will extend our illustration above from a single period to two periods, each with a riskless return rate equal to .25. As before, the stock currently sells for 10 and will change to either 15 or 5 in one time period (u = 1.5, d = .5). However, in the second period, the stock will change a second time by a factor of either 1.5 or .5, leading to potential values of either 22.5 (up then up again), 10 (up once and down once) or 2.5 (down twice). The lattice associated with this stock price process is depicted in Figure 1 below, and stock prices are listed with call option values in Figure 1 as well.

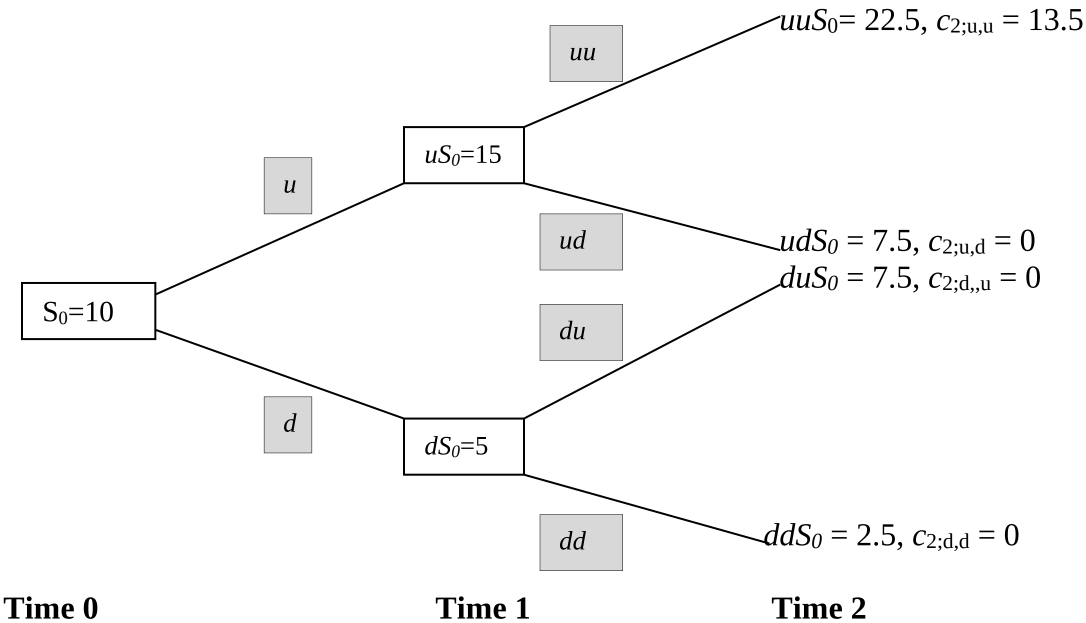

Figure 1: Two Period Binomial Model with Option Values

Since u, d and r are the same for each period, probability measure $\mathbb{Q}$ will be the same for each period. Thus, q = .75 and (1-q)= .25 in each each period. However, the hedge ratio α_t will change for each period, depending on the share price movement in the prior period:

$$\alpha_0 = \frac{c_{1;u} - c_{1;d}}{S_0(d-u)} = \frac{6-0}{10(.5-1.5)} = -.6$$

$$\alpha_{1,u} = \frac{c_{2;u,u} - c_{2;u,d}}{uS_0(d-u)} = \frac{13.5-0}{15(.5-1.5)} = -.9$$

$$\alpha_{1,d} = \frac{c_{2;d,u} - c_{2,d,d}}{dS_0(d-u)} = \frac{0-0}{7.5(.5-1.5)} = 0 \quad (\textit{no hedge})$$

Thus, the hedge ratio must be adjusted after each price change. Actually, there is no hedge or hedge ratio after the stock price decreases after time zero. The call option has no value in this event and cannot be used to hedge the stock risk. The hedge ratio for Time 1, assuming that the stock price increased after time zero will be -.9, meaning that .9 shares of stock must be short sold for each purchased call to maintain the hedged portfolio.

The time-zero 2-period binomial call price is calculated as follows:

$$c_0 = \frac{\sum_{j=0}^{2}\frac{2!}{j!(2-j)!}.75^{\,j}(1-.75)^{2-j}MAX[1.5^j.75^{2-j}\times10-9,0]}{(1+.25)^2} = \frac{c_{2;uu}\,q^{\,2}}{(1+r)^2} = \frac{13.5\times.5625}{1.5625} = 4.86$$

Notice that, in the two-period framework, the call has the same value at time zero (4.86) as 7.59375 bonds at time zero. In addition, its time zero expected value in time 2 is also the same as 7.59375 bonds:

$$E_{\mathbb{Q}}[c_2] = [c_{2;u,u}\,q^{\,2} + 2c_{2;u,d}q\,(1-q\,) + c_{2;d,d}\,(1-q\,)^2] = (22.5-9)\times.5625 = 7.59375$$

Thus, under the 2-period probability measure $\mathbb{Q}$ with the bond as the numeraire, the call price process is a martingale, just as the stock price process is:

$$E_{\mathbb{Q}}[S_{0,2}] = [u^2S_0q_u{}^2 + 2udS_0q\,(1-q\,) + d^2S_0(1-q\,)^2] = 15.625 = S_0/b_0,$$

where the current value of the stock in this two-time period framework is 15.625 times the time zero value (.64) of the bond.

<u>Extending the Model to Three Periods</u>

Now, we will extend our illustration above from two periods to three, each with a riskless return rate equal to .25. Two upjumps are necessary for the call to be exercised. By the third period, potential stock values are either $1.5^3 \times 10 = 33.75$, $1.5^2 \times .5 \times 10 = 11.25$, $.5^2 \times 1.5 \times 10 = 3.75$ or $.5^3 \times 10 = 1.25$. The time zero call value in this three-period binomial framework is computed with Equations 19 and 18 as follows:

$$a = MAX\left[INT\left(\frac{ln\left[\frac{9}{10\times.5^3}\right]}{ln\left[\frac{1.5}{.5}\right]}+1\right),0\right]=2$$

$$c_0 = \frac{[3\times.75^2\times.25\times(1.5^2\times.5\times10-9)]+[1\times.75^3\times(1.5^3\times10-9)]}{(1+.25)^3} = 5.832$$

Put-call parity still applies:[5]

$$p_0 = 5.832 + 9/(1.25)^3 - 10 = 0.44$$

C. Multiplicative Upward and Downward Movements in Practice

One apparent difficulty in applying the binomial model as it is presented above is in obtaining estimates for *u* and *d* that are required for *p*; all other inputs are normally quite easily obtained. There are several methods that are used to obtain parameters for the binomial method from the actual security returns generating process. For sake of simplicity here, we will assume that all investors are risk-neutral, and that physical probabilities and their martingale equivalents are the same ($p = q$). For example, following Cox, Ross and Rubinstein [1979] derive the following to estimate probabilities of an uptick *p* and downtick (1 - p):[6]

$$(20) \qquad p = \frac{e^{r_f} - d}{u - d} \qquad (1 - p) = \frac{u - e^{r_f}}{u - d}$$

Cox *et al.* also proposes the following to estimate *u* and *d* in the Binomial approximation to the Wiener process, where σ is the standard deviation of stock returns:

$$(21) \qquad u = e^{\sigma} \qquad d = \frac{1}{u}$$

or, if *n* and *T* differ from 1:

$$(22) \qquad u = e^{\sigma\sqrt{\frac{T}{n}}} \qquad d = \frac{1}{u}$$

Suppose, for example, that for a particular Wiener process, $\sigma = .30$ and $r_f = .05$. Using Equations 20 and 22 above, we estimate *p*, *u* and *d* for a single time period binomial process as follows:

$$u = e^{.3} = 1.3498588$$
$$d = \frac{1}{u} = .7408182$$

<u>The Binomial Model in Practice: An Illustration</u>

Suppose that we wished to evaluate a call and a put with an exercise price equal to \$110 on a share of stock currently selling for \$100. The underlying stock return standard deviation equals .30 and the current riskless return rate equals .05. If both options are of the European variety and expire in six months, what are their values?

First, we will compute the call's value using the binomial model. We will vary the number of jumps in the model as the example progresses. First, let $n = 1$ and use Equations 20

[5] We will discount the exercise money with a discrete discount function since the binomial model is a discrete time model.

[6] See Appendix A for derivations of *p*, *u* and *d*. In some of our illustrations, there will be a somewhat minor deviation from the probability estimates given by Equation set 20. The difference is likely that Equation set 20 allows interest (r_f) to be continuously compounded whereas we often use discrete compounding to calculate probabilities. This distinction is not important for our purposes here.

and 22 to compute *p*, *u* and *d*:

$$u = e^{.3\times\sqrt{.5}} = 1.2363111$$

$$d = \frac{1}{1.2363111} = .8088578$$

$$p = \frac{e^{r_f T} - d}{u - d} = 0.5063881$$

Thus, in a risk-neutral environment, there is a .5064 probability that the stock price will increase to 123.63 and a .4936 probability that the stock price will be 80.88678. Similarly, in a risk neutral environment, there is a .5064 probability that the call will be worth 13.63; therefore, its current value is 6.73 = .5064 × 13.63 × $e^{-.05\times.5}$. The call value is determined by the binomial model as follows where $n = a = 1$:[7]

$$c_0 = \frac{\sum_{j=a}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \left[u^j \times d^{n-j} S_o - X\right]}{\left(1 + r_f\right)^T}$$

$$c_0 = \frac{.506388^1 \times .4936119^{1-1} \times \left[1.236311^1 \times .8088578^{1-1} \times 100 - 110\right]}{\left(1 + .05\right)^{.5}} = 6.73$$

We can also use the binomial model to value the put with identical exercise terms on the underlying stock:[8]

$$p_0 = \frac{\sum_{j=0}^{a-1} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \left[X - u^j d^{n-j} S_o\right]}{\left(1 + r_f\right)^T}$$

$$p_0 = \frac{.506388^0 \times .4936119^1 \times \left[110 - 1.236311^0 \times .8088578^1 \times 100\right]}{\left(1 + .05\right)^{.5}} = 14.08$$

<u>Dividing an Interval into Sub-Intervals</u>

Now, divide the single six-month interval into two three-month intervals; that is, $n = 2$. We will now use a two-period binomial model to evaluate calls and puts on this stock. First, we

[7] *a* is determined by Equation 16 and is the first positive integer where $u^j d^{n-j} S_0 > X$. That is, the minimum number of up-jumps required for exercise of the call option equals *a*. Any smaller number of stock up-jumps produces a terminal call value equal to zero, and need not be considered.

[8] Recall that put-call parity, demonstrated in Section C, can be used to value puts. We verify this for this example by using Equation 3 as follows: 14.08 = 6.73 + 110$e^{-.05\times.5}$ - 100.

use Equations 20 and 22 to compute p, u and d:[9]

$$u = e^{..3\sqrt{\frac{.5}{2}}} = 1.1618342$$

$$d = \frac{1}{1.1618342} = .8607079$$

$$p = \frac{e^{r_f T/n} - d}{u - d} = .5043415$$

Thus, there is a $.5043^2$ probability that the stock price will increase to 134.98, a .5 probability that the stock price will remain unchanged at 100 and a $.4957^2$ probability that the stock price will decline to 74.08. Thus, there is a .2543 probability that the call will be exercised, in which case, it will be worth 24.98. Therefore, the call's current value is 6.20 = .2543 × .24.98 × $e^{-.05 \times .5}$. Call and put values are determined by the binomial model as follows where n = a = 2:

$$c_0 = \frac{\sum_{j=a}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \left[u^j d^{n-j} S_o - X\right]}{\left(1+r_f\right)^T}$$

$$c_0 = \frac{.5043^2 \times .4957^{2-2} \times \left[1.16185^2 \times .8607^{2-2} \times 100 - 110\right]}{\left(1+.05\right)^{.5}} = 6.20$$

$$p_0 = \frac{\sum_{j=0}^{a-1} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \left[X - u^j d^{n-j} S_o\right]}{\left(1+r_f\right)^T}$$

$$p_0 = \frac{.5043^0 \times .4957^2 \times \left[110 - 1.1618^0 \times .8607^2 \times 100\right] + 2 \times .5043 \times .4957 \times [110 - 100]}{\left(1+.05\right)^{.5}}$$
$$= 6.20 + 110e^{-.05 \times .5} - 100 = 13.48$$

As the 6-month period is divided into more and finer subintervals, the values of the call and put will approach their Black-Scholes values. Table 1 extends this example to more than two

[9] When there are multiple jumps per period (n > 1), and/or when T does not equal one, $p = \frac{e^{r_f (T/n)} - d}{u - d}$ and $\left(1-p\right) = \frac{u - e^{r_f (T/n)}}{u - d}$.

subintervals, ultimately approaching the Black-Scholes model.

n	c_0	p_0
1	6.73	14.02
2	6.20	13.48
3	5.47	12.72
4	6.04	13.30
5	5.18	12.44
6	5.91	13.17
7	5.43	12.68
8	5.81	13.06
9	5.57	12.82
10	5.73	12.98
50	5.63	12.89
100	5.59	12.85
∞	5.59	12.85

Volatility	$\sigma = .30$
Riskless rate	$r_f = .05$
Exercise price	$X = 110$
Initial stock price	$S_0 = 100$
Term to expiration	$t = .5$

Table 1: Convergence of the Binomial Model to the Black-Scholes Model

References

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Exercises

1. Examination of price data for a given stock reveals that the stock has a 41% probability of increasing by 42% in any given six-month period and a 59% probability of decreasing by 30% during any six-month period. The stock has a current market value equal to $60; a one-year call option exists on the stock with an exercise price equal to $50. Assume an annual riskless return rate equal to 5% and that it is compounded continuously.
   a. What is the probability that the stock's price will exceed $100 at the end of six months?
   b. What is the probability that the stock's price will exceed $100 at the end of one year? (One year represents two six-month intervals.)
   c. What are the two potential stock prices at the end of the first six-month period? What are the probabilities associated with each of these prices?
   d. What are the three potential stock prices at the end of the second six-month period? What are the probabilities associated with each of these prices?
   e. What are the three potential call option values at the end of one year? That is, what would be the value of the call option conditional on each of the three potential stock prices being realized?
   f. If the stock price increases during the first six-month period, two potential stock prices are possible at the end of one year. What are these two prices? What are the potential call values? Based on these two potential prices and associated probabilities, what would be the value of the call option if the stock price increases during the first six-month period? Note that we are pricing the call based upon the physical probabilities, so that we will not obtain risk neutral pricing.
   g. If the stock price decreases during the first six-month period, two potential stock prices are possible at the end of one year. What are these two prices? What are the potential call values? Based on these two potential prices and associated probabilities, what would be the value of the call option if the stock price decreases during the first six-month period?
   h. Based on the two potential call option values estimated in parts *f* and *g* and their associated probabilities, what is the current value of the call?

2. A stock currently selling for $100 has a 75% probability of increasing by 20% in each time period and a 25% chance of decreasing by 20% in a given period. Assume a binomial process. What is the expected value of the stock after four periods?

3. In We have claimed that in a multi-period binomial framework affecting stock prices, $c_0 = \frac{\sum_{j=0}^{T}\frac{T!}{j!(T-j)!}q^{\,j}(1-q\,)^{T-j}MAX[u^j d^{T-j}S_0 - X, 0]}{(1+r_f)^T}$. Derive this formula for call valuation. You may assume that the riskless hedge can be created and that valuation occurs in a riskless framework over T periods of time.

4. Consider the binomial returns process with upjump *u* and downjump *d* covered earlier. In that section, we claimed that if $pu + (1-p)d = 1$, then S_t is a martingale.
   a. Demonstrate that S_t is indeed a martingale under the circumstances given above.
   b. Suppose that the riskless bond prices are $B_0 = 1$ and $B_t = (1+r)^t$. Further assume that S_t follows the binomial returns process described above with probability *q* of an upjump.

Demonstrate that if $qu + (1- q)d = 1+r$, then S_t is a martingale with respect to the bond (change of numeraire).

c. Under the same circumstances as in part b above, demonstrate that if $q = (1+r-d)/(u-d)$, then S_t is a martingale with respect to the bond.

5. Assume that a binomial pricing model holds for all securities with a probability p of an upjump $1+u$ and a probability $1 - p$ of a downjump $1 - d$. Also suppose that a risk free rate of return r holds at each time interval.

a. Derive an expression that solves for the discounted expected value of the price S_n at time n given the price S_m at time m.

b. Find the value of p so that the discounted expected value satisfies $E[S_n|S_m]/(1+r)^{n-m}=S_m$ for $m<n$; that is, find the value of p so that S_n is a martingale (in bond numeraire).

6. Consider a one time period, two potential outcome framework where there exists Company Q stock currently selling for $50 per share and a riskless $100 face value T-Bill currently selling for $90. Suppose Company Q faces uncertainty, such that it will pay its owner either $30 or $70 in one year. Further assume that a call with an exercise price of $55 exists on one share of Q stock.

a. Define the Equivalent Probability Measure for this payoff space.
b. Is this measure an equivalent martingale (risk-neutral probability measure)?
c. Is this market complete?
d. What are the two potential values the call might have at its expiration?
e. What is the riskless rate of return for this example? Remember, the Treasury bill pays $100 and currently sells for $90.
f. What is the hedge ratio for this call option?
d. What is the current value of this option?
h. What is the value of a put with the same exercise terms as the call?

7. Rollins Company stock currently sells for $12 per share and is expected to be worth either $10 or $16 in one year. The current riskless return rate is .125.

a. Define the Equivalent Probability Measure for this payoff space.
b. Is this market complete?
c. What would be the value of a one-year call with an exercise price of $8?

8. The riskless return rate equals .08 per year in a given economy. A stock, which currently sells for $50, has an expected per-jump multiplicative upward movement for this stock is 1.444, and d = 1/u. Under the binomial framework, what would be the value of nine-month (.75 year) European calls and European puts with striking prices equal to $80 if the number of tree steps (n) over the 9-month period were:

a. 2
b. 3
c. 8

Assume the riskless return rate is compounded over each relevant time increment for parts a, b, and c.

9. Ibis Company stock is currently selling for $50 per share and has a multiplicative upward

movement equal to 1.2776 and a multiplicative downward movement equal to .7828. What is the value of a nine-month (.75 year) European call and a European put with striking prices equal to $60 if the number of tree steps were 2? Assume a riskless return rate equal to .081.

10. Show that in the binomial pricing model, the expectation:

$$E[S_t|S_s] = S_s[pu + (1-p)d]^{t-s}$$

for $s<t$ can be derived from the equation:

$$E[S_t|S_0] = S_0[pu + (1-p)d]^t.$$

Solutions

1. a. 0; The highest potential stock price equals \$85.2.
   b. 2 upjumps are required; $.41^2 = .1681$
   c. 1.42 × \$60 = \$85.2 with probability .41; .7 × \$60 = \$42 with probability .59
   d. 1.42^2 × \$60 = \$120.98 with probability $.41^2$=.1681; 1.42 × .7 × \$60 = \$59.64 with probability 2×.41×.59=.4838; $.7^2$ × \$60 = \$29.4 with probability $.59^2$=.3481.
   e. MAX[\$120.98-\$50,0] = \$70.98; MAX[\$59.64-\$50,0] = \$9.64; MAX[\$29.4-\$50,0] =0
   f. Potential stock prices are \$120.98 and \$59.64
      Potential call values are \$70.98 and \$9.64
      The discounted call value would be [.41×\$70.98+.59×9.64]$e^{-.05\times.5}$ = \$33.93
   g. Potential stock prices are \$59.64 and \$29.58
      Potential call values are \$9.64 and 0
      The discounted call value would be [.41×\$9.64+.59×0]$e^{-.05\times.5}$=\$3.85
   h. The current discounted call value is [.41×\$33.93+.59×\$3.85]$e^{-.05\times.5}$=\$15.78

2. The expected value of the stock is computed as follows:

$$E[S_4]=S_0[pu+(1-p)d]^t=[.75\times1.2+.25\times.8]^4=\$146.41.$$

3. The value of the call should be the expected value of the random variable $MAX[S_T - X,0]$ discounted by the risk-free rate of return. To compute the expected value, suppose there have been j upjumps to time T. The remaining T-j jumps would then be downjumps. The value of $MAX[S_T - X,0]$ in this case is $MAX[u^j d^{T-j} S_0 - X,0]$, and this occurs with probability $\binom{n}{j} q^j (1-q)^{T-j}$. So, the expected value equals:

$$E_{\mathbb{Q}}\big[MAX[S_T - X,0]\big] = \sum_{j=0}^{T} \binom{n}{j} q^j (1-q)^{T-j} MAX\big[u^j d^{T-j} S_0, 0\big].$$

With a risk-free rate of return or discount rate equal to r, the time T value will have been discounted by $(1 + r)^{-T}$. The result now follows.

4.a. We showed earlier that we have the conditional expectation:

$$E_{\mathbb{P}}[S_t|S_s] = [pu + (1-p)d]^{t-s} S_s$$

for $s<t$. If $pu + (1- p)d = 1$, then $[pu + (1- p)d]^{t-s} = 1$ and $E_{\mathbb{P}}[S_t|S_s] = 1 \times S_s = S_s$.
   b. From section 5.2.1, we know that $E_{\mathbb{Q}}[S_t|S_s] = [qu + (1-q)d]^{t-s} S_s$ for $s<t$. From time s to time t, we need to discount the expected value by the amount $(1+r)^{t-s}$. With respect to the bond, the stock expected value is

$$\frac{E_{\mathbb{Q}}[S_t|S_s]}{[1+r]^{t-s}} = \frac{[qu+(1-q)d]^{t-s} S_s}{[1+r]^{t-s}} = [\frac{qu+(1-q)d}{1+r}]^{t-s} S_s = 1 \times S_s = S_s$$

if $(qu+(1-q)d)/(1+r)=1$ or equivalently $qu+(1-q)d=1+r$. Thus, S_t is a martingale with respect to the bond in this case.
   c. $qu + (1- q)d = 1+r$. Rewrite as $qu+d-qd = 1+r$. Thus, $q(u-d)+d = 1+r$ and $q = (1+r-d)/(u-d)$.

5. a. Substituting 1+u in place of u, 1 – d in place of d, n in place of t, and m in place of s in the formula for the expected value of S_t given S_s at the end of section 5.2.1, we obtain:

$$E_P[S_n|S_s]=[p(1+u)+(1-p)(1-d)]^{n-m}S_m.$$

Since a risk free investment of $1 at time m will result in its value equaling $(1+r)^{n-m}$ at time n, then the discounted expected value of S_n at time m must be

$$\frac{E_{\mathbb{P}}[S_n|S_m]}{[1+r]^{n-m}} = \frac{[p(1+u)+(1-p)(1-q)]^{n-m}}{[1+r]^{n-m}}S_m$$
$$= \left[\frac{p(1+u)+(1-p)(1-q)}{1+r}\right]^{n-m} S_m.$$

b. To assure that S_n is a martingale, which means that $E[S_n|S_m]/(1+r)^{n-m} = S_m$, we require:

$$\frac{p(1+u)+(1-p)(1-d)}{1+r} = 1.$$

Solving for p gives

$$p = \frac{d+r}{u+d}.$$

6. a. Risk-neutral probabilities are computed as follows:

$$\begin{bmatrix} 30 & 70 \\ 100 & 100 \end{bmatrix}\begin{bmatrix} \psi_{0,1;d} \\ \psi_{0,1;u} \end{bmatrix} = \begin{bmatrix} 50 \\ 90 \end{bmatrix} \;;\; \begin{matrix} \psi_{0,1;d} = .325 \\ \psi_{0,1;u} = .575 \end{matrix}$$
$$q_{0,1;d} = \psi_{0,1;d}/(\psi_{0,1;d}+\psi_{0,1;u}) = .36111$$
$$q_{0,1;u} = \psi_{0,1;u}/(\psi_{0,1;d}+\psi_{0,1;u}) = .63889$$

b. Yes: There are no arbitrage strategies in this payoff space.

c. Yes: The Probability measure is unique because the number of priced securities equals the number of potential outcomes, and their set of payoff vectors is complete.

d. $c_T = MAX[0,S_T-X]$; c_T = $0 or $15

e. $100/$90 - 1 = .1111

f. The hedge ratio is computed as follows:

$$\alpha = \frac{c_u - c_d}{S_0(d-u)} \quad \alpha = \frac{15-0}{50(.6-1.4)} = -.375$$

$$c_0 = \frac{c_u q + c_d(1-q)}{1+r} = \frac{15\times.63889 + 0\times(1-.63889)}{1+.11111} = 8.625$$

g.

h. $$p_0 = \frac{p_u q + p_d(1-q)}{1+r} = \frac{0\times.63889 + 25\times(1-.63889)}{1+.11111} = 8.125$$

7. a. Risk-neutral probabilities are computed as follows:

$$\begin{bmatrix} 10 & 16 \\ 100 & 100 \end{bmatrix}\begin{bmatrix} \psi_{0,1;d} \\ \psi_{0,1;u} \end{bmatrix} = \begin{bmatrix} 12 \\ 88.888889 \end{bmatrix} \;;\; \begin{matrix} \psi_{0,1;d} = .37037 \\ \psi_{0,1;u} = .518519 \end{matrix}$$
$$q_{0,1;d} = \psi_{0,1;d}/(\psi_{0,1;d}+\psi_{0,1;u}) = .416667$$
$$q_{0,1;u} = \psi_{0,1;u}/(\psi_{0,1;d}+\psi_{0,1;u}) = .583333$$

b. Yes: There are no arbitrage opportunities, the probability measure is unique because the number of priced securities equals the number of potential outcomes, and their set of payoff vectors is complete.

c. This problem can be set up in either of two comparable ways:

First, find the hedge ratio:

$$\alpha = \frac{c_u - c_d}{S_0(d-u)}$$

$$\alpha = \frac{8-2}{12(.83333 - 1.3333)} = -1$$

Now, value the call:

$$c_0 = \frac{-(1+r)\alpha S_0 + C_d + \alpha d S_0}{1+r}$$

$$c_0 = \frac{-(1+.125)\cdot(-1)\cdot 12 + 2 + (-1)\cdot .83333 \cdot 12}{1+.125} = 4.8889$$

Alternatively, the risk neutral probabilities from above can be used:

$$c_0 = \frac{c_u q + c_d(1-q)}{1+r} = \frac{8 \times .58333 + 2 \times (1-.58333)}{1+.125} = 4.88889$$

8. First, we will estimate q_u from u, d (which equals 1/u = .69252) and r:

$$d = \frac{1}{1.444} = .69252; \; q_{u,2} = \frac{1+.08\times .75/2 - .69252}{1.444 - .69252} = 0.449087$$

$$q_{u,3} = \frac{1+.08\times .75/3 - .69252}{1.444 - .69252} = 0.435779; \; q_{u,8} = \frac{1+.08\times .75/8 - .69252}{1.444 - .69252} = 0.4191$$

For the 2-time period framework, call valuation calculations proceed as follows:

$$a = INT\left[MAX\left[\frac{\ln\left(\frac{80}{50\cdot .69252^2}\right)}{\ln\left(\frac{1.444}{.69252}\right)}, 0\right] + 1\right] = 2$$

$$c_0 = \frac{.449087^2 \times .550913^{2-2} \times \left[1.444^2 \times .69252^{2-2} \times 50 - 80\right]}{(1+.08\cdot .75/2)^2} = 4.61$$

For the 3-time period framework, call valuation calculations proceed as follows:

$$a = INT\left[MAX\left[\frac{\ln\left(\frac{80}{50\cdot .69252^3}\right)}{\ln\left(\frac{1.444}{.69252}\right)}, 0\right] + 1\right] = 3$$

$$c_0 = \frac{.435779^3 \times .564221^{3-3} \times \left[1.444^3 \times .69252^{3-3} \times 50 - 80\right]}{(1+.08\cdot .75/3)^3} = 5.50$$

For the 8-time period framework, call valuation calculations proceed as follows:

$$a = INT\left[MAX\left[\frac{\ln\left(\frac{80}{50\cdot .69252^8}\right)}{\ln\left(\frac{1.444}{.69252}\right)}, 0\right] + 1\right] = 5$$

$$c_0 = \frac{.4191^8 \times .5809^{8-8} \times \left[1.444^8 \times .6925^{8-8} \times 50 - 80\right]}{(1+.08\cdot .75/8)^8} + 8 \times \frac{.4191^7 \times .5809^{8-7} \times \left[1.444^7 \times .6925^{8-7} \times 50 - 80\right]}{(1+.08\cdot .75/8)^8}$$

$$+28\times\frac{.4191^{6}\times.5809^{8-6}\times\left[1.444^{6}\times.6925^{8-6}\times50-80\right]}{\left(1+.08\cdot.75/8\right)^{8}}+56\times\frac{.4191^{5}\times.5809^{8-5}\times\left[1.444^{5}\times.6925^{8-5}\times50-80\right]}{\left(1+.08\cdot.75/8\right)^{8}}$$

$$=14.39$$

Put values are found with put-call parity. The following are call and put values for the 2, 3 and 8 period frameworks:

n	c_0	p_0
2	4.61	30.02
3	5.50	30.89
8	14.39	39.75

The calls are easily valued in the 2- and 3-step models because the maximum number of upjumps (2 and 3) are required for exercise. In the 8-period framework, 5 to 8 upjumps were required for exercise.

9. $S_0 = 50$, $X = 60$, $T = 0.75$, $r = 0.081$, $u = 1.2776$, $d = 0.7828$

$$q_u = \frac{1+.081\times.75\times.5-.7828}{1.2776-.7828} = .5$$

$$a = INT\left[MAX\left(\frac{ln\left(\frac{60}{50\times.7828^2}\right)}{ln\left(\frac{1.2776}{.7828}\right)},0\right)+1\right] = 2$$

$$c_0 = \frac{.5^2\times.5^{2-2}[1.2776^2\times.7828^{2-2}\times50-60]}{(1+.081\times.75\times.5)^2} = \$5.09$$

$c_0 = \$5.09$; Based on put-call parity, $p_0 = -50+60(1+.081\times.75\times.5)^{-2} + 5.09 = \$11.60$

10. Simply regard time *s* as time 0, so that S_0 becomes the value S_s. Thus time *t* becomes time *t-s* since it takes time *t-s* to go from time *s* to time *t.* The value S_t remains S_t, since to go from the value S_s in *t-s* steps of time will give us S_t. So keeping S_t on the left side of the equation, replacing S_0 with S_s and *t* with *t-s* on the right side of the given equation gives the desired result.

Appendix A: The Binomial Model: Additional Considerations

A Computationally More Efficient Version of the Binomial Model

Here, we simply discuss a computationally more efficient version of the n-time period Binomial model equation 17 from the text:

(17) $$c_0 = \frac{\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} MAX[0, u^j d^{n-j} S_0 - X]}{\left(1+r_f\right)^n}$$

The number of computational steps required to solve Equation 17 is reduced if we eliminate from consideration all of those outcomes where the option's expiration date price is zero. Thus, the smallest non-negative integer for j where $S_n>X$ is the smallest integer that exceeds the following:

(19) $$a = MAX\left[\frac{\ln\left(\frac{X}{S_0 d^n}\right)}{\ln\left(\frac{u}{d}\right)}, 0\right]$$

This result is obtained by first determining the minimum number of price increases j needed for S_n to exceed X:

(A.1) $$S_n = u^j d^{n-j} S_0 > X$$

We then solve this inequality for the minimum non-negative integer value for j such that $u^j d^{n-j} S_0 > X$. Take logs of both sides to obtain:

(A.2) $$\begin{gathered} j \cdot \ln(u) + n \cdot \ln(d) - j \cdot \ln(d) + \ln(S_0) > \ln X \\ j \cdot \ln\left(\frac{u}{d}\right) > \ln(X) - n \cdot \ln(d) - \ln(S_0) \end{gathered}$$

Next divide both sides by *ln*(u/d) and simplify to obtain Equation 19, where a is the smallest non-negative integer for j for which $S_n>X$: Finally, we substitute Equation 19 into Equation 7 to obtain *The Binomial Option Pricing Model*:

(18) $$c_0 = \frac{\sum_{j=a}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \left[u^j d^{n-j} S_o - X\right]}{\left(1+r_f\right)^n}$$

or:

$$c_0 = S_0\left[\sum_{j=a}^{n}\frac{n!}{j!(n-j)!}\frac{(pu)^j\left[(1-p)d\right]^{n-j}}{\left(1+r_f\right)^n}\right] - \frac{X}{\left(1+r_f\right)^n}\left[\sum_{j=a}^{n}\frac{n!}{j!(n-j)!}p^j(1-p)^{n-j}\right] \quad \text{(18.a)}$$

or, in shorthand form:

$$c_0 = S_0 B_1[a,n,p'] - X\left(1+r_f\right)^{-n} B_2[a,n,p] \quad \text{(18.b)}$$

where $p' = pu/(1+r_f)$. There are two more points regarding equation 18b. First, assuming that investors behave as though they are risk neutral, B_2[a, n, p] can be interpreted as the probability that the stock price will be sufficiently high at the expiration date of the option to warrant its exercise. Second, B_2[a, n, p '] can be interpreted as a hedge ratio, though it must be updated at every period.

Obtaining Multiplicative Upward and Downward Movement Values

One apparent difficulty in applying the binomial model as it is presented above is in obtaining estimates for u and d that are required for p; all other inputs are normally quite easily obtained. There are several methods that are used to obtain parameters for the binomial method from the actual security returns generating process. For example, following Cox, Ross and Rubinstein [1979], we can begin the process of estimating the mean and variance to be used in the binomial distribution by first approximating the mean and variance for the binomial process from the historical Wiener process as follows:

$$E\left[S_T\right] = S_0 e^{\mu T + \frac{1}{2}\sigma^2 T} \approx puS_0 + (1-p)dS_0 \quad (1)$$

$$\sigma_s^2 T = S_0^2\left(e^{\sigma^2 T} - 1\right)\left(e^{2\mu T + \sigma^2 T}\right) \approx \left[pu^2 S_0^2 + (1-p)d^2 S_0^2\right] - \left[puS_0 + (1-p)dS_0\right]^2 \quad (2)$$

Approximation 2 approaches equality as T approaches zero. Scaling S_0 to one such that we work with returns rather than actual security prices, the following can be used for returns variance of a binomial process:

$$\sigma^2 = \left[pu^2 + (1-p)d^2\right] - \left[pu + (1-p)d\right]^2 \quad (3)$$

We will rewrite Equation 3 as follows:[10]

$$\sigma^2 = p(1-p)(u-d)^2 \quad (4)$$

[10] The following are algebraic steps to obtain Equation 4 from Equation 3:

$$\sigma^2 = \left[pu^2 + (1-p)d^2\right] - \left[pu + (1-p)d\right]^2 = \left[pu^2 + d^2 - pd^2\right] - \left[p^2u^2 + (1-p)^2 d^2 + 2pu(1-p)d\right]$$
$$\sigma^2 = \left[pu^2 + d^2 - pd^2\right] - \left[p^2u^2 + d^2 + p^2d^2 - 2pd^2 + 2pud - 2p^2ud\right]$$
$$\sigma^2 = \left[pu^2\right] - \left[p^2u^2 + p^2d^2 - pd^2 + 2pud - 2p^2ud\right]$$
$$\sigma^2 = p(u^2 - pu^2 - pd^2 + d^2 - 2ud + 2pud) = p(1-p)(u^2 + d^2 - 2ud)$$

Assume that the binomial process will lead to expected return for a security equaling the riskless rate:

(5) $$e^{\mu T+\frac{1}{2}\sigma^2 T} = e^{r_f} \approx pu + (1-p)d$$

Solving Equation 5 for p enables us to estimate probabilities of an uptick p and downtick (1 - p) as:

(6) $$p = \frac{e^{r_f} - d}{u-d} \qquad (1-p) = \frac{u - e^{r_f}}{u-d}$$

If we define d as $1/u$ such that $ud = 1$, we can rewrite Equation 4, the variance of returns, as follows:

(7) $$\sigma^2 = p(1-p)(u-d)^2 = p(1-p)\left(e^{\delta} - e^{-\delta}\right)^2$$

Thus, we have simply substituted some constant e^{δ} for u. Substituting σ for δ will normally provide a good approximation for variance (improving as the number of jumps in the binomial process, n approaches infinity):

(8) $$\sigma^2 = p(1-p)\left(e^{\delta} - e^{-\delta}\right)^2 = p(1-p)\left(e^{\sigma} - e^{-\sigma}\right)^2$$

Thus, we can use the following to estimate u and d in the Binomial approximation to the Wiener process:

(9) $$u = e^{\sigma} \qquad d = \frac{1}{u}$$

or, if n and T differ from 1:

(10) $$u = e^{\sigma\sqrt{\frac{T}{n}}} \qquad d = \frac{1}{u}$$

Suppose, for example, that for a particular Wiener process, $\sigma = .30$ and $r_f = .05$. Using equations (5) and (8) above, we estimate p, u and d for a single time period binomial process as follows:

$$u = e^{.3} = 1.3498588$$
$$d = \frac{1}{u} = .7408182$$

We can verify our estimates with Equations (4) and (7) as follows:

$$e^{.005+\frac{1}{2}\times .3^2} = e^{.05}$$

$$= .5097409 \times 1.3498588 + (1 - .5097409) \times .7408182 = 1.0512$$

$$\sigma^2 = p(1-p)\left(e^{\delta} - e^{-\delta}\right) \approx .5097409 \times (1 - .5097409)\left(e^{.3} - e^{-.3}\right)^2 = .0926974$$

As discussed above, there are several procedures for getting parameters σ, u, d and p for the binomial distribution. This procedure is probably the most commonly used, in part, because it provides a relatively straightforward method for estimating option Greeks. The primary difficulty with the one presented above is that it may result in negative probabilities. An additional methodology for estimating binomial distribution parameters is given in Jarrow and Turnbull [1996], pp. 136-38.