Chapter 11: The Greeks, Dividend Adjustments and Early Exercise

A. The Greeks

Levels of sensitivity of the Black-Scholes model (See Black and Scholes [1972]) to each of its 5 inputs are colloquially known as the *Greeks*. The first of these Greeks is the option *delta*, or $N(d_1)$, which is the instantaneous sensitivity of the option value to changes in the stock price:[1]

$$\frac{\partial c}{\partial S} = N(d_1) > 0 \qquad\qquad \text{Delta} \quad \Delta$$

This sensitivity means that the call option's value will change by approximately $N(d_1)$ for each unit of change in the underlying stock price. This delta can be interpreted as the number of shares to short for every purchased call in order to maintain a portfolio that is hedged to changes in the underlying stock price. A portfolio delta is simply a weighted average (weights are provided by portfolio proportions) of the option deltas in the portfolio. A delta-neutral portfolio means that the portfolio of options and underlying stock has a weighted average delta equal to zero so that its value is invariant with respect to the underlying stock price.

However, as the underlying stock's price changes over time, and, as the option's time to maturity diminishes, this hedge ratio will change; the option delta only holds exactly for an instant. That is, because this delta is based on a partial derivative with respect to the share price, it holds exactly only for an infinitesimal change in the share price; it holds only approximately for finite changes in the share price. This delta only approximates the change in the call value resulting from a change in the share price because any change in the price of the underlying shares would lead to a change in the delta itself:

$$\frac{\partial^2 c}{\partial S^2} = \frac{\partial \Delta}{\partial S} = \frac{\partial N(d_1)}{\partial S_0} = \frac{e^{\left(-\frac{d_1^2}{2}\right)}}{\sqrt{2\pi T}\, S_0 \sigma} > 0 \qquad\qquad \text{Gamma} \quad \Gamma$$

This change in delta resulting from a change in the share price is known as *gamma*. However, again, this change in delta resulting from a finite share price change is only approximate.

Since each call and put option has a date of expiration, calls and puts are said to amortize over time. As the date of expiration draws nearer (T gets smaller), the value of the European call (and put) option might be expected to decline as indicated by a positive *theta*:[2]

$$\frac{\partial c}{\partial T} = rXe^{-rT} N(d_2) + \frac{S_0 \sigma}{2\sqrt{2\pi T}} e^{-\frac{d_1^2}{2}} > 0 \qquad\qquad \text{Theta} \quad \theta$$

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[1] Appendix B to this chapter derives the delta and gamma expressions.

[2] This expression is derived in end-of-chapter Exercise 7.11.a. More importantly, its relation to the all-important Black-Scholes differential equation 11 in Section 7.3.2 is described in Exercise 7.11.c. Many traders refer to theta by its negative value, emphasizing that the value of the option decays through the passage of time. Theta is also known as amortization. Many traders will also divide the annual theta by 252, the number of trading days in a year in order to reflect the daily amortization.

By convention, traders often refer to theta as a negative number since option values tend to decline as we move forward in time (*T* becomes smaller). In addition, many traders seek to maintain portfolios that are simultaneously neutral with respect to delta, gamma and theta.

Vega, which actually is not a Greek letter, measures the sensitivity of the option price to the underlying stock's standard deviation of returns (vega is sometimes known as either *kappa* or *zeta*). One might expect the call option price to be directly related to the underlying stock's standard deviation:

$$\frac{\partial c}{\partial \sigma} = \frac{S_0\sqrt{T}}{\sqrt{2\pi}} e^{\left(-\frac{d_1^2}{2}\right)} > 0 \qquad \textit{Vega } \nu$$

Although the Black-Scholes model assumes that the underlying stock volatility is constant over time, in reality, as we discussed in the previous section, volatility can and does shift. Vega provides an estimate for the impact of a volatility shift on a particular option’s value.

A trader should expect that the value of the call would be directly related to the riskless return rate and inversely related to the call exercise price:

$$\frac{\partial c}{\partial e^r} = TXe^{-rT} N(d_2) > 0 \qquad \textit{Rho } \rho$$

$$\frac{\partial c}{\partial X} = -e^{-rT} N(d_2) < 0$$

The option rho can be very useful in economies with very high or volatile interest rates, though most traders of “plain vanilla" options (standard options to buy or sell without complications) do not concern themselves much with it under typical interest rate regimes. Similarly, most traders of "plain vanilla" options ignore call value sensitivities to exercise prices.

<u>Greeks Calculations for Calls</u>

Here, we will calculate the Greeks for the call illustration from the previous section 7.4 with the following parameters:

$T = .5$ $\quad r = .10$ $\quad \sigma = .41147$ $\quad X = 80$ $\quad S_0 = 75$ $\quad c_0 = 8.20$

$$\frac{\partial c}{\partial S} = \Delta = N(d_1) = N(.0955) = .538$$

$$\frac{\partial^2 c}{\partial S^2} = \Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial N(d_1)}{\partial S} = \frac{e^{\left(\frac{-d_1^2}{2}\right)}}{S_0\sigma\sqrt{2\pi T}} = \frac{e^{\left(\frac{-.0955^2}{2}\right)}}{75 \times .41147 \times \sqrt{\pi}} = .0182$$

$$\frac{\partial c}{\partial T} = \theta = rXe^{-rT}N(d_2) + S\frac{\sigma}{\sqrt{T}}\frac{e^{\left(\frac{-d_1^2}{2}\right)}}{2\sqrt{2\pi}} = .1\times 80e^{-.05}N(-.1954) + 75\frac{.41147}{\sqrt{.5}}\frac{e^{\left(\frac{-.0955^2}{2}\right)}}{2\sqrt{2\pi}} = 11.882$$

$$\frac{\partial c}{\partial \sigma} = \nu = \frac{S_0\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} = \frac{80\times\sqrt{.5}}{\sqrt{2\pi}}e^{-\frac{0955^2}{2}} = 21.06$$

$$\frac{\partial c}{\partial e^r} = \rho = TXe^{-rT}N(d_2) = .5\times 80\times e^{-.05}N(-.1954) = 16.077$$

Note that traders sometimes change the sign for theta, actually using the derivative ∂c/∂t where *T* is replaced by (*T*-*t*) and *t* represents time as it approaches *T*. More generally, a single unit increase in the relevant parameter changes the Black-Scholes estimated call value by an amount approximately equal to the associated "Greek."

Greeks Calculations for Puts

If Put-Call Parity holds along with Black-Scholes assumptions given above, the Black-Scholes put value from our example in Section 7.5.1 is computed as follows:

$$p_0 = -S_0N(-d_1) + \frac{X}{e^{rT}}N(-d_2) = -75N(-.0955) + \frac{80}{e^{.05}}N(.1954) = 9.298$$

In our numerical example above, the put would be worth 9.298 if its exercise terms are identical to those of the call. Put sensitivities formulas and calculations for our example are as follows:[3]

$$\frac{\partial p}{\partial S} = \Delta_p = N(d_1) - 1 = .538 - 1 = -.462$$

$$\frac{\partial^2 p}{\partial S^2} = \Gamma_p = \frac{\partial \Delta_p}{\partial S} = \frac{\partial(N(d_1)-1)}{\partial S} = \frac{e^{\left(\frac{-d_1^2}{2}\right)}}{S_0\sigma\sqrt{2\pi T}} = \frac{e^{\left(\frac{-.0955^2}{2}\right)}}{75\times .41447\sqrt{2\pi\times .5}} = .0182$$

$$\frac{\partial p}{\partial T} = \theta_p = rXe^{-rT}N(-d_2) - \frac{S_0\sigma}{2\sqrt{2\pi T}}e^{-\frac{d_1^2}{2}} = .1\times 80\times e^{-.05}N(.1954) - \frac{75\times .41147}{2\sqrt{\pi}}e^{-\frac{0.0955^2}{2}} = -4.27$$

$$\frac{\partial p}{\partial \sigma} = \nu_p = \frac{S_0\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} = \frac{75\times\sqrt{.5}}{\sqrt{2\pi}}e^{-\frac{0955^2}{2}} = 21.06$$

[3] The put delta and gamma are derived in Exercise 7.10.

$$\frac{\partial p}{\partial e^{r}} = \rho_p = -TXe^{-rT}N(-d_2) = -.5\times 80e^{-.05}N(.1954) = -21.97$$

Again, a single unit increase in the relevant parameter changes the Black-Scholes estimated put value by an amount approximately equal to the associated "Greek."

B. The Black-Scholes Model and Dividend Adjustments

The Black-Scholes model has an enormous number of extensions. Here, we will first consider stock options in scenarios in which certain standard Black-Scholes assumptions are violated. Relaxation of assumptions allows the model to be applied to options written under varying circumstances and will have a number interesting implications. Here, we will discuss a few of the many extensions of the model, beginning with adjustments for stock dividends.

The European Known Dividend Model

A *dividend-protected* call option allows for the option holder to receive the underlying stock and any dividends paid during the life of the option in the event of exercise. In practice, most options are not dividend protected. In effect, a dividend payment diminishes the value of the underlying stock by the value of the dividend on the ex-dividend date.[4] If a stock underlying such a European call option were to pay a known dividend of amount D, with ex-dividend date t_D $< T$, the Black-Scholes hedge portfolio and differential equation become:

$$c_t = \gamma_{s,0}(S_0 - De^{-rt_D}) + \gamma_{b,0}B_{0,}$$

$$\frac{\partial c}{\partial t} = rc - r(S_0 - De^{-rt_D})\frac{\partial c}{\partial S} - \frac{1}{2}\sigma^2 S_0^2\frac{\partial^2 c}{\partial S^2}$$

With the boundary condition $c_T = MAX\left[0,(S_T - De^{r(T-t_D)} - X)\right]$, the *European Known Dividend Model* can be used to evaluate the option as follows:

$$c_0 = c_0\left[S_0 - De^{-rt_D}, T, r, \sigma, X\right] = \left(S_0 - De^{-rt_D}\right)N(d_1) - \frac{X}{e^{rT}}N(d_2)$$

$$d_1 = \frac{\ln\left(\dfrac{S_0 - De^{-rt_D}}{X}\right) + \left(r + \dfrac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

This is among the simplest of the dividend-adjusted Black-Scholes models. Multiple dividend payments produce the boundary condition $c_T = MAX\left[0,(S_T - \sum_{t_{Di}=1}^{n} D_i e^{r(T-t_{Di})} - X)\right]$, where the firm makes a dividend payments to shareholders at each of n points in time t_{Di} prior to time T.

[4] A shareholder holding the stock on the ex-dividend date receives the dividend. Shareholders obtaining the stock after the ex-dividend date do not. Also, when underlying stock returns volatility is computed, dividends are excluded from the calculations.

Thus, generally, the analyst simply subtracts the present value of the known dividend payment (or payments) prior to the option expiry from the stock's price. However, this model still assumes that the call is of the European variety, so that the call will never be exercised early. Nevertheless, holders of non-protected calls will not receive dividends if they obtain the underlying shares after the ex-dividend date. If this model should be used to value American calls on dividend-paying stocks, it will tend to undervalue these American calls that cannot be exercised on ex-dividend dates.

Modeling American Calls

An American call on a non-dividend paying stock will never be exercised before its expiration date *T*. Consider Table 10.1, which depicts a scenario in which an investor can choose whether to exercise his American call now prior to its expiration at time *T*. In effect, the investor can choose between two portfolios *A* and *B* where Portfolio *A* consists of one call that is not exercised before expiry and the present value of *X* dollars Xe^{-rT} retained from not exercising the call. If the call is exercised before expiry, the portfolio, labeled as Portfolio B, will consist of one share of stock, which is purchased with the exercise money *X*. The last two columns of this table give expiration date portfolio values, depending on the underlying stock price S_T relative to the call exercise price *X*. Since the sum value of the call and exercise money is always either equal to or greater than the value of the stock on the expiration date ($V_{AT} \geq V_{BT}$), the call should never be exercised early. That is, Portfolio *A* is always preferred to Portfolio *B*.

However, Table 2 demonstrates that premature exercise of an American call might occur when its underlying stock pays a dividend, where t_D ($t_D \leq$ T) is the time of the premature exercise. We will demonstrate shortly that this premature exercise will occur only on the ex-dividend date, and only when the dividend amount is sufficiently high relative to some critical ex-dividend date stock price S_{tD}^*.

Portfolio Value at Time T

Portfolio	Current value	$S_T \leq X$	$S_T > X$
A	$c_0 + Xe^{-rT}$	0 + X	$(S_T - X) + X$
B	S_0	S_T	S_T
Note that:		$V_{AT} > V_{BT}$	$V_{AT} = V_{BT}$

Table 10.1: Exercising American Calls Early in the Absence of Dividends

Portfolio Value at Time T

Port.	Current Value	$S_T > X$	$S_T \leq X$
A	$c_0 + Xe^{-rT}$	$S_T - X + X$	X
B	$S_0 + De^{-rt_D}$	$S_T + De^{r(T-t_D)}$	$S_T + De^{r(T-t_D)}$
	Notice that:	$V_{AT} \leq V_{BT}$	$V_{AT} \gtrless V_{BT}$

Table 2: Exercising American Calls Early in the Presence of Dividends

Black's Pseudo-American Call Model

This model (Black [1975]) incorporates the European Known Dividend model with the choice of early exercise. Recall that an American option should never be exercised if the stock pays no dividend. Similarly it can be shown that if an American call is to be exercised early, it will be exercised only at the instant (or immediately before) the stock goes ex-dividend. If, on the ex-dividend date t_D, the time value associated with exercise money, $X[1-e^{-r(T-t_D)}]$ is more than the dividend payment, D, the call will never be exercised early. Otherwise early exercise is optimal when the dividend is sufficiently large. Black's model states that the call's value c_0 is determined by:

$$c_0 = MAX\left[c_0^*, c_0^{**}\right]$$
$$c_0^* = c_0^*\left[S_0, t_D, r, \sigma, X\right]$$
$$c_0^{**} = c_0^{**}\left[S_0 - De^{-rt_D}, T, r, \sigma, X\right]$$

where c_0^* is the call's value assuming it is exercised immediately before the stock's ex-dividend date and c_0^{**} is the call's value assuming the option is held until it expires. Observe from the formula that the call's actual value, c_0, will be the larger of c_0^* or c_0^{**}. Since the call's value is the larger of c_0^* and c_0^{**}, and these values are both determined at time zero, the formula implies that the American call value is based on an exercise decision at time zero. That is, even though we do not decide whether to exercise early until the ex-dividend date, the formula implies that the call value is based on a decision today on whether to exercise on the ex-dividend date.

Illustration: Calculating the Value of an American Call on Dividend-Paying Stock

Suppose that we wish to calculate the value of a 6-month X = $45 American call on a share of stock that is currently selling for $50. The stock is expected to go ex-dividend on a $5 payment in three months, and not again until after the option expires. The standard deviation of returns on the underlying stock is .4 and the riskless return rate is 3%. What is the call worth today assuming:

a. the European Known Dividend model?
b. Black's Pseudo-American Call model?

a. Under the European Known Dividend model, the value of the call is $5.39, calculated as follows:

$$c_0 = (50 - 5e^{-.03\times.25}) \times .5782 - \frac{45}{e^{.03\times.5}} \times .4660 = 5.39,$$

where

$$d_1 = \frac{\ln\left(\frac{50 - 5e^{-.03\times.25}}{45}\right) + \left(.03 + \frac{1}{2}\times.16\right)\times.5}{.4\times\sqrt{.5}} = .1974; N(d_1) = .5782$$

$$d_2 = .1974 - .4\times\sqrt{.5} = -.0855; N(d_2) = .4660$$

b. To use Black's Pseudo-American call formula, we will first determine whether dividend exceeds the ex-dividend date time value associated with exercise money: $5 > 45[1-e^{-.03(.5-.25)}] = .336$. In other words, we check to see if we would lose more than \$5 on interest from our exercise money. Since the dividend does exceed this value, we will calculate the value of the call assuming that we exercise it just before the underlying stock goes ex-dividend in 3 months, while it trades with dividend:

$$c_0 = 50 \times .7468 - \frac{45}{e^{.03\times.25}} \times .6788 = 7.02,$$

where

$$d_1 = \frac{\ln\left(\frac{50}{45}\right) + \left(.03 + \frac{1}{2}\times .16\right)\times .25}{.4\times\sqrt{.25}} = .6643; N(d_1) = .7468$$

$$d_2 = .6643 - .4\times\sqrt{.25} = .4643; N(d_2) = .6788$$

Under Black's Pseudo-American call formula, we find that the value of the call is *MAX*[5.39, 7.02], or 7.02.

C. Merton's Continuous Leakage Formula

In some instances, dividends might be considered to be paid on a continuous basis. For example, many options on indices (index options) trade on portfolios whose *dividend leakage*, the rate at which continuous dividends are paid or received by a fund, can often be accounted for when assumed to occur continuously and without detectable seasonality given the time to option expiry. Other options can be traded on commodities or other assets that have costs of storage or other constant carry costs that are paid continuously over time. Here, we assume that the underlying stock follows the process below:

$$\frac{dS_t}{S_t} = (\mu - \delta)dt + \sigma dZ_t$$

where δ is the periodic dividend yield (See Merton [1973]). The self-financing portfolio in this case has the form:

$$dV_t = \gamma_{s,t}dS_t + r\left(V_t - \gamma_{s,t}S_t\right)dt + \delta\gamma_{s,t}S_t dt.$$

The standard Black-Scholes differential equation for the continuous leakage model is as follows:

$$\frac{\partial c}{\partial t} = rc - (r-\delta)\frac{\partial c}{\partial S}S - \frac{1}{2}\sigma^2 S_0^2\frac{\partial^2 c}{\partial S^2}$$

This is the continuous dividend adjusted Black-Scholes differential equation. Its particular solution, subject to the *boundary condition* c_T = MAX[0, S_T - X] for a European call, is given as follows:

$$c_0 = S_0 e^{-\delta T} N(d_1) - \frac{X}{e^{rT}} N(d_2),$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$p_0 = c_0 + Xe^{-rT} - S_0 e^{-\delta T}$$

Illustration: Continuous Dividend Leakage

Return to our earlier example where an investor is considering six-month options on a stock that is currently priced at $75. The exercise price of the call and put are $80 and the current riskless rate of return is 10% per annum. The variance of annual returns on the underlying stock is 16%. However, this stock will pay a continuous annual dividend at a rate of 2%. What are the values of these call and put options on this stock? Our first step is to find d_1 and d_2:

$$d_1 = \frac{\ln\left(\frac{75}{80}\right) + \left(.10 - .02 + \frac{1}{2} \times .16\right) \times .5}{.4\sqrt{.5}} = .0547$$

$$d_2 = .0547 - .4\sqrt{.5} = -.2282$$

With $N(d_1)$ = .522 and $N(d_2)$ = .41, we value the call and put as follows:

$$c_0 = 75e^{-.02 \times .5} \times .522 - \frac{80}{e^{.10 \times .5}} \times .41 = 7.56$$

$$p_0 = 7.56 + 80(.9512) - 75e^{-.02 \times .5} = 9.41$$

D. Early Option Exercise

First, it is important to note that an American call on a non-dividend paying stock should never be exercised before its expiration date. Consider portfolios A and B in Table 3, where Portfolio A consists of one call that is not exercised before expiry and the present value of X dollars Xe^{-rfT} retained from not exercising the call. If the call is exercised at expiry, the portfolio consists of one share of stock. Portfolio B consists of a single share of the underlying stock:

Portfolio	Current value	$S_T \leq X$	$S_T > X$
A	$c_0 + X_e^{-r_1 T}$	0 + X	$(S_T - X) + X$
B	S_0	S_T	S_T
Note that:		$V_{AT} > V_{BT}$	$V_{AT} = V_{BT}$

Table 3: Portfolio Time T Value without Stock Dividends

Since the sum value of the call and exercise money is always either equal to or greater than the value of the stock on the expiration date ($V_{AT} > V_{BT}$), the call should not be exercised early. That is, Portfolio A is preferred to Portfolio B.

However, premature exercise (i.e., exercise prior to expiry) of an American call may be rational when its underlying stock pays a dividend of *D* that is sufficiently high. Consider the following:

Port.	Current Value	$S_T > X$	$S_T \leq X$
A	$C_0 + X$	$S_T - X + Xe^{r_f T}$	$Xe^{r_f T}$
B	$S_0 + D$	$S_T + De^{r_f T}$	$S_T + De^{r_f T}$
	Notice that:	$V_{AT} \underset{>}{\overset{<}{-}} V_{BT}$	$V_{AT} \underset{<}{\overset{>}{-}} V_{BT}$

Table 4: Portfolio Time *T* Value with Stock Dividends

In Portfolio A, the investor retains his call and exercise money. He will exercise his call when it expires at time *T* if $S_T > X$ and earn the riskless return on his exercise money *X* regardless of whether S_T is less or greater than *X*. In Portfolio B, he exercises his call on the ex-dividend date, receiving the dividend that may be invested at the riskless rate. Whether he exercises his call early (on the ex-dividend date - or the instant before ex-dividend) depends on the size of the dividend. An extremely large dividend is likely to encourage early exercise. The call will never be exercised early when $D < X[1-e^{-rT}]$. Generally, the call will be exercised early when its price on the stock ex-dividend date (S_{ex}) is less than the current stock price minus the call exercise price: $c_{ex} < S_{ex} - X$. In this scenario, just before the stock goes ex-dividend, owning the stock after paying the exercise price is worth more than the unexpired call. Regardless, an American call on a dividend paying stock will be exercised either at its expiration or the instant before the call goes ex-dividend.

In practice, American puts tend to be exercised more frequently than American calls. Early exercise of an American put is rational if the exercise money to be received at exercise exceeds the present value of the put left unexpired. Deep in the money puts are more likely to be exercised early. High interest rates and low volatilities are more likely to lead to early put exercise.

References

Black, Fischer (1975). "Fact and Fantasy in the Use of Options," *Financial Analysts Journal* 31, pp. 36-41, 61-72.

Black, Fischer and Myron Scholes (1972). "The Valuation of Options Contracts and a Test of Market Efficiency," *Journal of Finance*, 27, pp. 399-417.

Black, Fischer and Myron Scholes (1973): "The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, vol. 81, pp. 637-654.

Canina, L. and Figlewski, S. (1993). "The informational content of implied volatility," *Review of Financial Studies*, 6, pp. 659–681.

Cox, John C., Steven Ross and Mark Rubinstein (1979). "Option Pricing: A Simplified Approach," *Journal of Financial Economics* 7, pp. 229-263.

Drezner, Z. (1978). "Computation of the Bivariate Normal Integral," *Mathematics of Computation* 32, pp. 277-279.

Garman, M. and S. Köhlagen. "Foreign Currency Option Values," *Journal of International Money and Finance*, December, 1983.

Geske, R. (1979a). "The valuation of compound options," *Journal of Financial Economics*, 7, pp. 63-81.

Geske, R. (1979b). "A note on an analytic valuation formula for unprotected American call options on stocks with known dividends," *Journal of Financial Economics*, 7, pp. 375-380.

Harrison, J.M. and D.M. Kreps (1979). "Martingales and arbitrage in multi-period securities markets," *Journal of Economic Theory*, pp. 381-408.

Harrison, J.M. and S. Pliska (1981). "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Processes and Their Applications* 11, pp. 215-260.

Hull, John C. (2010). *DerivaGem, version 2.01*, online software package, http://www.rotman.utoronto.ca/~hull/software/bivar.xls. Accessed 08/17/2014.

Hull, John C. (2011). *Futures, Options and Other Derivatives*, 8th ed. Englewood Cliffs, New Jersey: Prentice Hall.

Latane, Henry and Richard Rendleman (1976). "Standard Deviations of Stock Price Ratios Implied by Option Prices," *Journal of Finance* 31: pp. 369-381.

Merton, Robert C. (1973). "Theory of Rational Option Pricing," *Bell Journal of Economics and*

Management Science, 4 (1), pp. 141-183.

Roll, R. (1977). "An analytic valuation formula for unprotected American call options on stocks with known dividends," *Journal of Financial Economics*, 5, pp. 251-258.

Whaley, R. E. (1981). "On the valuation of American call options on stocks with known dividends," *Journal of Financial Economics*, 9, pp. 207-211.

Whaley, R. E. (1982). "Valuation of American call options on dividend-paying stocks: Empirical tests," *Journal of Financial Economics*, 10, pp. 29-58.

Exercises

1. The sensitivity of the call's price to changes in the underlying stock return standard deviation is often known as "vega," which is calculated from the following:

$$\frac{\partial c}{\partial \sigma} = \frac{S_0\sqrt{T}}{\sqrt{2\pi}} e^{\frac{-d_1^2}{2}} > 0 \qquad \textit{Vega } \nu$$

Derive this equation above for vega, using the identity:

$$S_0 \frac{\partial N(d_1)}{\partial d_1} = Xe^{-rT} \frac{\partial N(d_2)}{\partial d_2}$$

that is derived in Appendix B to this chapter.

2. What is the gamma of a long position in a single futures contract? Why?

3. In the chapter, we stated the delta and gamma for a put. Derive both.

4. a. As the time to maturity T of a call option increases, the call value will increase. Find and simplify the derivative of the Black Scholes option pricing model c_0 with respect to time to maturity (T). As we discussed in the chapter, this derivative is often known as the option Theta. Hint: Make use of the product and chain rules from calculus and pay close attention to the simplification procedure described in Appendix B to this chapter.
   b. As the call option approaches maturity, its value will diminish. That is, as we move forward through time (t), the value of the call will tend to decline as its expiration draws closer. Based on part a of this problem, find the rate of change of the value for the call with respect to time.
   c. The Black-Scholes options pricing model is derived from the Black-Scholes differential equation:

$$\frac{\partial c}{\partial t} = rc - r\frac{\partial c}{\partial S}S - \frac{1}{2}S^2\sigma^2\frac{\partial^2 c}{\partial S^2}$$

   Using your results from part b of this problem, verify that the Black-Scholes pricing model is a valid solution to the Black-Scholes differential equation.

5. Tiblisi Company stock currently sells for 30 per share and has an anticipated volatility (annual return standard deviation) equal to .6. Three-month (.25 year) call options are available on this stock with an exercise price equal to 25. the current riskless return rate equals .05.
   a. Calculate the call's value.
   b. Calculate the call's delta, gamma, theta, vega and rho.
   c. What is the Black-Scholes implied probability that the stock price will exceed 25 three months from now?
   d. What is the value of a put with the same exercise terms as the call?
   e. What are the Greeks for the put in part d?

6. Consider two-year options on a stock that is currently priced at \$20. The exercise price of the call and put are \$10 and the current riskless rate of return is 4% per annum. The standard deviation of annual returns on the underlying stock is .8. However, this stock will pay a

continuous annual dividend at a rate of 5%. What are the values of these call and put options on this stock assuming continuous dividend leakage?

7. Consider a 6-month option on an index contract. The index is currently priced at 34,500 and the 6-month call has an exercise price of 35,000. The current riskless rate of return is 10% per annum. However, the stocks comprising the index pay dividends, resulting in a 3% continuous annual dividend payout by the portfolio underlying the index. The 6-month option currently sells for 4139.86. What is the standard deviation (within 2%) of annual returns on this index contract consistent with the call market price?

8. An investor has the opportunity to purchase a six-month call option for $2.00 on a stock that is currently selling for $30. The stock is expected to go ex-dividend in 3 months on a declared dividend of $2. The exercise price of the call is $35 and the current riskless rate of return is 3% per annum. The variance of annual returns on the underlying stock is 16%.

a. Based on the European Known Dividend Model, and its current price of $2.00, does this option represent a good investment?

b. What is the value of a European put on this stock?

c. Value the American call using Black's Pseudo-American call model.

d. Value the American call using the Geske-Roll-Whaley compound call model.

9. Owners of many commodities must make payments to store and maintain their inventories. In addition, many of these inventories suffer from depreciation or depletion. Suppose that the combined costs of storage and depletion for a given amount of a particular commodity (say a bushel of a particular variant of corn) is qS_t, where q is the proportional storage and depletion cost per unit of the commodity per unit time.

a. Write the Black-Scholes differential equation defining the option price path for this unit of corn.

b. Write a variation of the Black-Scholes option pricing model for a call on a single bushel of corn.

Solutions

1. Differentiating the call function with respect to σ and using the chain rule we have:

$$\frac{\partial c}{\partial \sigma} = S_0 \frac{\partial N(d_1)}{\partial \sigma} - Xe^{-rT} \frac{\partial N(d_2)}{\partial \sigma} = S_0 \frac{dN(d_1)}{d(d_1)} \frac{\partial d_1}{\partial \sigma} - Xe^{-rT} \frac{dN(d_2)}{d(d_2)} \frac{\partial d_2}{\partial \sigma}.$$

Since $d_1 = d_2 + \sigma\sqrt{T}$, then

$$\frac{\partial d_1}{\partial \sigma} = \frac{\partial d_2}{\partial \sigma} + \sqrt{T}.$$

This gives:

$$\frac{\partial c}{\partial \sigma} = S_0 \frac{dN(d_1)}{d(d_1)} \left(\frac{\partial d_2}{\partial \sigma} + \sqrt{T}\right) - Xe^{-rT} \frac{dN(d_2)}{d(d_2)} \frac{\partial d_2}{\partial \sigma}$$

$$= S_0\sqrt{T} \frac{dN(d_1)}{d(d_1)} + \frac{\partial d_2}{\partial \sigma}\left(S_0 \frac{dN(d_1)}{d(d_1)} - Xe^{-rT} \frac{dN(d_2)}{d(d_2)}\right) = \frac{S_0\sqrt{T}}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2}$$

and we are done.

2. Zero. This is because the futures contract can be replicated with a long position in a call and a short position in a put with the same exercise terms. The gammas of the call and the put are the same. A long position in a futures contract is replicated with a single long position in a call and a single short position in a put. Thus, the gamma of the long call position offsets the gamma of the short put position. We can also solve this problem by computation. By the call-put parity we have the futures contract has the value $V_0 = c_0 - p_0 = S_0 - Xe^{-rT}$. Thus the gamma of this portfolio equals:

$$\frac{\partial^2 V_0}{\partial S_0^2} = 0.$$

3. Since by put-call parity $p = c - S + Xe^{-rT}$, $\frac{\partial p}{\partial S} = \frac{\partial c}{\partial S} - 1 = \Delta_c - 1 = N(d_1) - 1$. This means that

$$\frac{\partial^2 p}{\partial S^2} = \frac{\partial \Delta_p}{\partial S} = \frac{\partial (N(d_1) - 1)}{\partial S} = \frac{1}{S\sigma\sqrt{2\pi T}} e^{-\frac{d_1^2}{2}} = \Gamma_p$$

4.a. The Black-Scholes model is written as follows:

$$c_0 = S_0 N(d_1) - Xe^{-rT} N(d_2).$$

We will use a combination of the product and chain rules to differentiate c_0 with respect to T to obtain the option theta:

$$\theta = \frac{\partial c_0}{\partial T} = S_0 \frac{d(N(d_1))}{d(d_1)} \frac{\partial d_1}{\partial T} - Xe^{-rT} \frac{d(N(d_2))}{d(d_2)} \frac{\partial d_2}{\partial T} - Xe^{-rT} N(d_2)(-r).$$

We will rewrite this derivative to exploit a trick describe in Appendix 7B to this chapter that will enable us to group and eliminate some terms afterwards. Since $d_1 = d_2 + \sigma\sqrt{T} = d_2 + \sigma T^{\frac{1}{2}}$, then $\frac{\partial d_1}{\partial T} = \frac{\partial d_2}{\partial T} + \frac{1}{2}\sigma T^{-\frac{1}{2}} = \frac{\partial d_2}{\partial T} + \frac{\sigma}{2\sqrt{T}}$. Putting this result into the equation above and simplifying slightly gives:

$$\frac{\partial c_0}{\partial T} = S_0 \frac{d(N(d_1))}{d(d_1)} \left[\frac{\partial d_2}{\partial T} + \frac{\sigma}{2\sqrt{T}}\right] - Xe^{-rT} \frac{d(N(d_2))}{d(d_2)} \frac{\partial d_2}{\partial T} + rXe^{-rT} N(d_2)$$

$$\frac{\partial c_0}{\partial T} = \left[S_0 \frac{d\big(N(d_1)\big)}{d(d_1)} - Xe^{-rT}\frac{d\big(N(d_2)\big)}{d(d_2)}\right]\frac{\partial d_2}{\partial T} + \frac{S_0\sigma}{2\sqrt{T}}\frac{d\big(N(d_1)\big)}{d(d_1)} + rXe^{-rT}N(d_2).$$

Using equation (34) in Appendix 7.B shows that the expression in the square brackets above equals zero, and so:

$$\theta = \frac{\partial c_0}{\partial T} = \frac{S_0\sigma}{2\sqrt{2\pi T}}e^{-\frac{1}{2}d_1^2} + rXe^{-rT}N(d_2).$$

b. As we derived in Appendix 7.A, the solution for the call at time *t* is given by equation (33):

$$c = SN\big(d_1(S, T-t)\big) - Xe^{-r(T-t)}N\big(d_2(S, T-t)\big),$$

where the functions

$$d_1(S, T-t) = \frac{ln\left(\frac{S}{X}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2(S, T-t) = d_1(S, T-t) - \sigma\sqrt{T-t}.$$

Notice that the price *c* of the call at time *t* is the same as the price of the call at time 0, $c_0(S_0,T)$, if one replaces S_0 with *S* and *T* with $T - t$ in the expression for c_0. Essentially, it is as if time *t* becomes time 0, so that *S* is now the initial stock price and the time to reach the exercise date is $T - t$. This can be expressed as: $c = c_0(S, T-t)$. Thus, it must be the case by the chain rule that:

$$\frac{\partial c}{\partial t} = \frac{\partial c_0(S, T-t)}{\partial (T-t)}\frac{d(T-t)}{dt} = -\frac{\partial c_0(S, T-t)}{\partial (T-t)} = -\theta(S, T-t).$$

Using part a, we obtain the required derivative:

$$\frac{\partial c}{\partial t} = \frac{-S\sigma}{2\sqrt{2\pi(T-t)}}e^{-\frac{1}{2}d_1^2} - rXe^{-r(T-t)}N(d_2)$$

where d_1 and d_2 are evaluated at *S* and $T - t$.

b. In order to show that the Black-Scholes pricing equation we obtained for the call is a solution of the Black- Scholes differential equation, we first substitute $c - SN(d_1)$ for $-Xe^{-r(T-t)}N(d_2)$ into the right side of the equation for $\frac{\partial c}{\partial t}$ above and rearrange terms as follows:

$$\frac{\partial c}{\partial t} = r\big(c - SN(d_1)\big) - \frac{1}{2}S^2\sigma^2\frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} \cdot \frac{1}{S\sigma\sqrt{T-t}}.$$

As derived in Appendix B to this chapter, we have the following equalities:

$$\frac{\partial c}{\partial S} = N(d_1)$$

$$\frac{\partial^2 c}{\partial S^2} = \frac{e^{\left(\frac{-d_1^2}{2}\right)}}{\sqrt{2\pi}} \cdot \frac{1}{S_0\sigma\sqrt{T}}$$

that applied at time 0. If we consider these values at time *t*, then:

$$N(d_1) = \frac{\partial c}{\partial S}$$

and

$$\frac{\partial^2 c}{\partial S^2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{S\sigma\sqrt{T-t}}$$

where d_1 is evaluated at *S* and $T - t$. Substituting these equalities into the theta equation (actually its negative) above produces the Black-Scholes differential equation, which verifies that the Black Scholes model is a solution to the differential equation:

$$\frac{\partial c}{\partial t} = rc - rS\frac{\partial c}{\partial S} - \frac{1}{2}S^2\sigma^2\frac{\partial^2 c}{\partial S^2}$$

The essential difference between the two is the sign change, arising from the fact that a decrease in *T* means that the option expires sooner and an increase in *t* means that the option expires sooner; $\Theta=-\partial c/\partial t$. More importantly, this result verifies that the Black Scholes model is a solution to the Black-Scholes differential equation 23 in Section 7.3.2.

5. a. Tiblisi Company calls expiring in three months with an exercise price equal to 25 are currently worth 6.5725. This value is based on d_1 = 0.799405189, d_2 = 0.499405189, $N(d_1)$ = 0.787972249 and $N(d_2)$ = 0.691253018.

b. Greeks for the call are computed as follows:
Delta = 0.787972249
Gamma = 0.032206303
Theta = 6.070521207
Vega = 4.347657115
Rho = 4.266663344

c. $N(d_2)$ = 0.691253018

d. p_0 = 1.261959101

e. Delta = -0.212027751
Gamma = 0.032206303
Theta = -4.83605
Vega = 4.347657115
Rho = -1.905697909

6. Our first step is to find d_1 and d_2:

$$d_1 = \frac{\ln\left(\frac{20e^{-.05\times.2}}{10}\right) + \left(.04 + \frac{1}{2}\times.8^2\right)\times.2}{.8\sqrt{2}} = 1.16$$

$$d_2 = 1.16 - .8\sqrt{2} = .0293$$

We value the call and put as follows:

$$c_0 = 20e^{-.05\times.2}\times.877 - \frac{10}{e^{.04\times2}}\times.512 = 11.15$$

$$p_0 = 11.15 + 10(.923) - 20e^{-.04\times2} = 2.28$$

7. **σ = .4**: Because we are working with a European call with a continuous dividend, we should use the Merton Continuous Dividend model to value this index call. Our model inputs are c_0 = \$4139.86, S_0 = 34,500, δ = .03, T = .5, X = 35,000 and r_f = .10. Through a process of trial and iteration, we find that the implied volatility is **σ = .4**. For example, try an initial guess for standard deviation equal to, for example, σ=.3. This results in a call value equal to 3201. This call value is much too small, so increase the standard deviation estimate. Try a much larger estimate**,** say σ = .5. This estimate results in a call value that is too large at 5074. So, we try a smaller estimate. Ultimately, we arrive at an estimate within .02 of the correct standard deviation of **.4**, plus or minus .02. To find the implied volatility of the index contract, our first step is to find d_1 and d_2: based on (here, with correct) trial values:

$$d_1 = \frac{ln\left(\frac{34500}{35000}\right)+\left(.10+\frac{1}{2}.\mathbf{4}^2\right)\times.5}{.4\sqrt{.5}} = .2143$$

$$d_2 = -.2143 - .\mathbf{4}\sqrt{.5} = -.0686$$

$N(d_1) = .5848$; $N(d_2) = .4727$

$$c_0 = \frac{34500}{e^{.03\times.5}}\times.5848 - \frac{35000}{e^{.10\times.5}}\times.4727 = 4139.858$$

8. a. First, we work through the European known-dividend call model:

$$d_1 = \frac{\ln\left(\frac{30-2e^{.03\times.25}}{35}\right)+\left(.03+\frac{1}{2}\times.16\right)\times.5}{.4\times\sqrt{.5}} = -.5926; N(d_1) = .2767$$

$$d_2 = -.5926 - .4\times\sqrt{.5} = -.0875; N(d_2) = .1907$$

$$c_0 = (30-2e^{.03\times.25})\times.2767 - \frac{35}{e^{.03\times.5}}\times.1907 = 1.18$$

Since the call is currently selling for $2, it is not a good investment; it should be sold.

b. Based on put-call parity and the known European dividend model, the put is worth $p_0 = 1.18 + 35(.985) - 30+2e^{-.03\times.25} = 7.64$.

c. If the call were presumed to be exercised on its ex-dividend date, it would be worth .869:

$$d_1 = \frac{\ln\left(\frac{30}{35}\right)+\left(.03+\frac{1}{2}\times.16\right)\times.25}{.4\times\sqrt{.25}} = -.6333; N(d_1) = .2633$$

$$d_2 = -.6333 - .4\times\sqrt{.25} = -.8333; N(d_2) = .2024$$

$$c_0 = 30\times.2633 - \frac{35}{e^{.03\times.25}}\times.2024 = .869$$

Thus, the value of the call is MAX[1.18,.869] = 1.18.

d. Under the Geske-Roll-Whaley Model, we first calculate S_{tD}^*, the critical stock value on the ex-dividend date required for early exercise of the call:

$$c_{tD}^* = S_{tD}^* + D - X$$

$$4.38954 = 37.38954 + 2 - 35$$

The value of the American call is found to be 1.23, calculated as follows:

$$d_1 = \frac{\left[\ln\left(\frac{30-2\cdot e^{-.03\times.25}}{35}\right)+\left(.03+\frac{1}{2}\cdot.16\right)\cdot.5\right]}{.4\cdot\sqrt{.5}} = -.5926$$

$$d_2 = -.5926 - .4\cdot\sqrt{.5} = -.8754$$

$$y_1 = \frac{\left[\ln\left(\frac{30-2\cdot e^{-.03\times.25}}{37.38954}\right)+\left(.03+\frac{1}{2}\cdot.16\right)\cdot.25\right]}{.4\cdot\sqrt{.25}} = -1.3058$$

$$y_2 = -1.3058 - .4\cdot\sqrt{.25} = -1.5058$$

$$c_o = \left(30-2\cdot e^{-.03\times.25}\right)\cdot.09581 + \left(30-2\cdot e^{-.03\times.25}\right)\cdot.1994$$
$$-35\cdot e^{-.03\times.5}\cdot.1415 - (35-2)\cdot e^{-.03\times.25}\cdot.06606 = 1.23$$

9. a. The Merton Continuous leakage model applies in this case, so that:

$$\frac{\partial c}{\partial t} = r\,c - (r\ - q)\frac{\partial c}{\partial S}S - \frac{1}{2}S^2\sigma^2\,\frac{\partial^2 c}{\partial S^2}$$

b.

$$c_0 = S_0 e^{-qT} N(d_1) - \frac{X}{e^{r\,T}} N(d_2)$$

$$d_1 = \frac{\ln\left(S_0\,\frac{e^{-qT}}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Appendix 10.A: Deriving Black-Scholes Delta and Gamma

Black-Scholes and Delta

The Black-Scholes model is given by the following:

$$c_0 = S_0 N(d_1) - Xe^{-rT} N(d_2)$$

$$d_1 = \frac{ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{ln\left(\frac{S_0}{X}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

where N(d*) is the cumulative normal distribution function for (d*).

Option traders find it very useful to know how the values of their option positions will change as the various factors used in the pricing model change. For example, the sensitivity of the call's value to the stock's price is given by Delta:

$$\frac{\partial c}{\partial S} = N(d_1) > 0 \qquad\qquad Delta \quad \Delta$$

Applying the Chain Rule to Derive Delta

We obtain this delta by first applying the Chain Rule to the Black-Scholes model as follows:

$$\frac{\partial c}{\partial S} = N(d_1) + S_0 \frac{dN(d_1)}{d(d_1)} \frac{\partial d_1}{\partial S} - Xe^{-rT}\left(\frac{dN(d_2)}{d(d_2)} \frac{\partial d_2}{\partial S}\right)$$

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{S_0} \cdot \frac{1}{\sigma\sqrt{T}}$$

$$\frac{\partial c}{\partial S} = N(d_1) + \frac{1}{S_0} \cdot \frac{1}{\sigma\sqrt{T}}\left[S_0 \frac{dN(d_1)}{d(d_1)} - Xe^{-rT} \frac{dN(d_2)}{d(d_2)}\right]$$

Our expression for Delta is obtained once we demonstrate that the two terms inside the brackets offset each other:[5]

(34) $$S_0 \frac{dN(d_1)}{d(d_1)} = Xe^{-rT} \frac{dN(d_2)}{d(d_2)}$$

[5] This trick to eliminate offsetting terms is key to finding many of the option Greeks (sensitivities).

We re-write the right side of this equality as follows:

$$Xe^{-rT}\frac{dN(d_2)}{d(d_2)} = Xe^{-rT}\frac{dN\left(d_1 - \sigma\sqrt{T}\right)}{d(d_1)} = Xe^{-rT}\frac{1}{\sqrt{2\pi}}\cdot e^{\frac{-(d_1-\sigma\sqrt{T})^2}{2}}$$

We continue to re-write as follows:

$$Xe^{-rT}\frac{1}{\sqrt{2\pi}}e^{\frac{-(d_1-\sigma\sqrt{T})^2}{2}} = Xe^{-rT}\frac{1}{\sqrt{2\pi}}e^{\frac{-d_1{}^2}{2}}\cdot e^{\frac{-(-2d_1\sigma\sqrt{T}+\sigma^2 T)}{2}} = Xe^{-rT}\frac{dN(d_1)}{d(d_1)}e^{d_1\sigma\sqrt{T}}\cdot e^{\frac{-\sigma^2 T}{2}}$$

Substituting for d_1 from the Black Scholes model and then canceling offsetting terms, we have

$$Xe^{-rT}\frac{dN(d_1)}{d(d_1)}e^{d_1\sigma\sqrt{T}}\cdot e^{\frac{-\sigma^2 T}{2}} = Xe^{-rT}\frac{dN(d_1)}{d(d_1)}e^{\ln(S_0/X)+(r+.5\sigma^2)T}\cdot e^{\frac{-\sigma^2 T}{2}} = X\frac{dN(d_1)}{d(d_1)}e^{\ln(S_0/X)}$$

$$X\frac{dN(d_1)}{d(d_1)}e^{\ln(S_0/X)} = \frac{dN(d_1)}{d(d_1)}S_0$$

Now, it should be clear that equation (34) and the formula for delta are true:

$$S_0\frac{dN(d_1)}{d(d_1)} = Xe^{-rT}\frac{dN(d_2)}{d(d_2)} \text{ and } \frac{\partial c}{\partial S} = N(d_1)$$

<u>Gamma</u>

We can derive gamma, the derivative of delta with respect to *S* using the chain rule as follows:

$$\frac{\partial^2 c}{\partial S^2} = \frac{\partial N(d_1)}{\partial S} = \frac{dN(d_1)}{d(d_1)}\frac{\partial(d_1)}{\partial S} = \Gamma$$

$$\frac{\partial^2 c}{\partial S^2} = \frac{e^{\frac{-d_1^2}{2}}}{\sqrt{2\pi}}\cdot\frac{1}{S_0\sigma\sqrt{T}} = \Gamma$$

Appendix 10.B: Deriving Vega

Sources of sensitivity of the Black-Scholes model (See Black and Scholes [1972]) to each of its 5 inputs are known as the *Greeks*. For example, option prices are very sensitive to the risk σ of the underlying security. *Vega*, which actually is not a Greek letter, measures the sensitivity of the option price to the underlying stock's standard deviation of returns (vega is sometimes known as either *kappa* or *zeta*). Vega is calculated by finding the partial derivative of c_0 with respect to σ in the Black Scholes option pricing model. One might expect the call option price to be directly related to the underlying stock's standard deviation:

$$\frac{\partial c}{\partial \sigma} = \frac{S_0 \sqrt{T}}{\sqrt{2\pi}} e^{\left(-\frac{d_1^2}{2}\right)} > 0 \qquad \textit{Vega } \nu$$

Although the Black-Scholes model assumes that the underlying stock volatility is constant over time, in reality, volatility can and does shift. Vega provides an estimate for the impact of a small volatility shift on a particular option's value. For example, in our illustration in Section E, we can calculate the option vega as follows:

$$\frac{\partial c}{\partial \sigma} = \frac{200 \times \sqrt{2}}{\sqrt{2\pi}} e^{\left(-\frac{d_1^2}{2}\right)} = 92.958$$

This vega implies that a small increase in σ (e.g., .01, from .6904 to .7004) would result in an approximate change 92.958% as large in option value (e.g., from 83.196 to 83.97):

$$c_1 = c_0 + \nu \Delta\sigma = 83.196 + 92.958 \times .01 = 83.97$$

Vega can be used in a banking context to calculate the impact of a change in asset volatility on equity value. Vega-based calculations are more accurate for smaller changes in volatility.