Chapter 12: Beyond Plain Vanilla Options on Stock

A. Compound Options

The Black-Scholes differential equation and model have a huge number of widely varying applications, particularly when it is extended by relaxing assumptions. Relaxation of assumptions has many useful applications and interesting implications. Here, we will discuss a few of the many extensions of the model.

A compound option (See Roll [1977] and Geske [1979a and 1979b]) is simply an option on an option. While such options are useful in their own right as traded securities, their analyses have a variety of other applications as well. For example, limited liability equity gives shareholders the right to either pay off creditors in full and take control of the firm's assets or abandon their claim on the firm's assets. This limited liability corporate feature essentially results in equity being a call option on the firm's assets. Hence, a call option on limited liability equity is a compound option, a call on a call. American options have compound option features. For example, an American call might be exercised early just prior to the underlying stock's ex-dividend date. Alternatively, the owner of the call has the right to hold the call to its expiry, creating a call on a call, where the compound call is exercised by not exercising the American option just prior to the ex-dividend date. There are also a variety of real option applications (options on corporate securities and projects) for compound option pricing models.

A compound option has two exercise prices and two expiration dates, the first of each (X_1, T_1) that applies to the right to buy or sell the underlying option and a second of each (X_2, T_2) that applies to buying or selling the security that underlies the underlying option with $T_1 < T_2$. We will assume that the underlying option is a European option with exercise price X_2 and expiration date T_2. Denote the values of the underlying call and put at time t by $c_{u,t}$ and $p_{u,t}$, respectively. Assume that there is a European option on this underlying option with exercise price X_1 on expiration date T_1. Compound options come in 4 varieties with the following exercise date T_1 payoff functions, where $c_{u,T1}$ and $p_{u,T1}$ are exercise date underlying call and put values:

1. Call on call: MAX[$c_{u,T1}$ -X_1, 0]
2. Put on call: MAX[X_1 - $c_{u,T1}$, 0]
3. Call on put: MAX[$p_{u,T1}$ -X_1, 0]
4. Put on put: MAX[X_1 - $p_{u,T1}$, 0]

First, consider a call on a call. The compound call gives its owner the right to purchase an underlying call with value $c_{u,T1}$ at time T_1 for price X_1. This compound call is exercised at time T_1 only if the underlying call at time T_1 is sufficiently valuable, which will be the case if the stock underlying the underlying call is sufficiently valuable. That is, the call on the underlying call option is exercised at time T_1 only if the stock price S_{T1} exceeds some critical underlying stock price S^*_{T1} at the time. If the stock price S_{T1} exceeds this critical value S^*_{T1}, the underlying call value $c_{u,T1}$ will exceed the exercise price X_1 of the compound call. Thus, we first calculate this critical value S^*_{T1}, which solves:

$$c_{u,T1}(S^*_{T1}, T_1; r, T_2, X_2, \sigma) = X_1$$

To calculate the left side of this equation, we would use the Black Scholes model equation (24), replacing S_0 with the variable S^*_{T1}, X with the number X_2, and T with the number $T_2 - T_1$. The numbers r and σ would remain the same. We will demonstrate shortly how to iterate to find the value of S_{T1}^* at time T_1 that would lead to optimal exercise of the compound call given the first exercise price X_1. There are several numerical and iterative techniques for solving this equality for S_{T1}^* (e.g., the methods of bisection and Newton Raphson), but simple substitution and iteration will work.

Estimating Exercise Probabilities

Following the reasoning from Section 7.D, and assuming that the underlying stock price process is as we discussed in Chapter 7, the probability that the stock price at time T_1 will exceed the critical value S_{T1}^* is $N(d_2)$, calculated as follows:

$$d_1 = \frac{\left[ln\left(\frac{S_0}{S_{T1}^*}\right) + \left(r + \frac{1}{2}\sigma^2\right)T_1\right]}{\sigma\sqrt{T_1}}$$

$$d_2 = d_1 - \sigma\sqrt{T_1},$$

where $N(d_2)$ is the cumulative normal density function for d_2.

Exercising the underlying call requires that the stock price S_{T1} at time T_1 exceed its critical value S_{T1}^* *and* that the stock price S_{T2} exceeds the underlying option exercise price X_2 at time T_2. In other words, for both the compound call and its underlying call to be exercised, the value of the underlying stock must exceed S_{T1}^* at time 1 (that is, the value of the underlying call must exceed X_1) and the value of the underlying stock must exceed X_2 at time T_2. The probability of both occurring equals:

$$M\left(d_2, y_2; \sqrt{\frac{T_1}{T_2}}\right)$$

where:

$$y_1 = \frac{\left[ln\left(\frac{S_0}{X_2}\right) + \left(r + \frac{1}{2}\sigma^2\right)T_2\right]}{\sigma\sqrt{T_2}}$$

$$y_2 = y_1 - \sigma\sqrt{T_2}.$$

and that the correlation coefficient between returns during our overlapping exercise periods equals $\rho = \sqrt{T_1/T_2}$.[1] The bivariate (multinomial) normal distribution function $M(*,**;\rho)$ provides the joint probability distribution function that is used to calculate the probability that our two random variables, S_{T1} and S_{T2} exceed the time T_1 critical value S_{T1}^* and the time T_2 exercise price

[1] The processes are perfectly correlated during their overlapping period T_1 and are independent over their non-overlapping period T_2-T_1. Thus, T_1/T_2 is the proportion of variability in one random variable explained by the other and the r-square value between the two process is T_1/T_2. Thus, the correlation coefficient is $\rho = \sqrt{T_1/T_2}$.

X_2, respectively. A condensed version of table for this distribution is provided as an appendix at the end of this chapter.

Valuing the Compound Call

Relying on and extending the logic presented in Section 7.D (See Whaley [1981 and 1982]), we can derive the value of a European compound call with exercise price X_1 and expiration date T_1, on an underlying call with exercise price X_2 and expiration date T_2 on a share of stock with current price S_0 with the following result:

$$c_{0,call} = S_0 M\left(d_1, y_1; \sqrt{\frac{T_1}{T_2}}\right) - X_2 e^{-rT_2} M\left(d_2, y_2; \sqrt{\frac{T_1}{T_2}}\right) - X_1 e^{-rT_1} N(d_2),$$

where

$$d_1 = \frac{\left[ln\left(\frac{S_0}{S^*_{T1}}\right) + \left(r + \frac{1}{2}\sigma^2\right)T_1\right]}{\sigma\sqrt{T_1}}$$

$$d_2 = d_1 - \sigma\sqrt{T_1}$$

$$y_1 = \frac{\left[ln\left(\frac{S_0}{X_2}\right) + \left(r + \frac{1}{2}\sigma^2\right)T_2\right]}{\sigma\sqrt{T_2}}$$

$$y_2 = y_1 - \sigma\sqrt{T_2}$$

Illustration: Valuing the Compound Call

Consider a compound call with exercise price X_1 = 3, expiring in three months (T_1 = .25) on an equity call, with exercise price X_2 = 45, expiring in 6 months (T_2 = .5). The stock currently sells for S_0 = 50 and has a return volatility equal to σ = .4. Thus, the compound call gives its owner the right to buy an underlying call in three months for \$3, which will confer the right to buy the underlying stock in six months for \$45. The riskless return rate equals r = .03. What is the value of this compound call?

First, we calculate the critical value S^*_{T1} for underlying option exercise at time T_1. A search process reveals that this critical value equals 43.58191:

$$c_{u,T1} = S_{T1}^* \times N\left(\frac{\left[ln\left(\frac{S_{T1}^*}{45}\right) + \left(.03 + \frac{1}{2} \times .16\right) \times (.5 - .25)\right]}{.4 \times \sqrt{.5 - .25}}\right) - \frac{45}{e^{.03\times(.5-.25)}} N\left(\frac{\left[ln\left(\frac{S_{T1}^*}{45}\right) + \left(.03 - \frac{1}{2} \times .16\right) \times (.5 - .25)\right]}{.4 \times \sqrt{.5 - .25}}\right) = X_1 = 3; \quad S_{T1}^* = 43.58191$$

We obtained this critical value by a process of substitution and iteration, assuming that the underlying call would be purchased for X_1 = \$3 and would confer the right to buy the underlying stock three months later for X_2 = \$45. The minimum acceptable value justifying the underlying call's exercise is $c_{u,T1} = X_1$ = \$3, which means that the underlying stock price must be at least S_{T1}^* = 43.58191 at time T_1 to exercise the right to purchase the underlying call.

Note that the correlation coefficient between our two random variables is $\rho = \sqrt{.25/.5}$ = .7071. We will use the reasoning from Section 5.4 and the bivariate normal distribution along with the formulas in Section 7.6.2 to calculate the compound call. But first, we make a few intermediate calculations:

$$d_1 = \frac{\left[ln\left(\frac{50}{43.58191}\right) + \left(.03 + \frac{1}{2} \times .16\right) \times .25\right]}{.4\sqrt{.25}} = .8244$$

$$d_2 = .8244 - .4 \times \sqrt{.25} = .6244$$

$$y_1 = \frac{\left[ln\left(\frac{50}{45}\right) + \left(.03 + \frac{1}{2} \times .16\right) \times .5\right]}{.4 \times \sqrt{.5}} = .567$$

$$y_2 = .567 - .4 \times \sqrt{.5} = .2841$$

Finally, we calculate the time zero value of the compound call as follows:[2]

$$\begin{aligned} c_{0,call} &= 50\times \mathrm{M}(.8244, .567; .7071) - 45e^{-(.03\times.5)}\mathrm{M}(.6244, .2841; .7071) \\ &\quad - 3e^{-(.03\times.25)}N(.6244) \\ &= 50 \times .6529 - 45 \times e^{-(.03\times.5)} \times .5507 - 3e^{-(.03\times.25)} \times .7338 = 6.0465 \end{aligned}$$

Thus, we find that the value of this compound call equals 6.0465. The bivariate probabilities were calculated using a spreadsheet-based multinomial cumulative distribution calculator (See Drezner [1978] and Hull [2010] and [2011] for details). There is a N(d_2) = .7338 probability that the compound call will be exercised to purchase the underlying call and a .6118 probability that

[2] See Section 2.7.3 of Knopf and Teall [2018] for the procedure to calculate bivariate normal distributions. Alternatively, it might be easy to use the workbook for Chapter 8 created for this course, the spreadsheet for the Cumulative Standard Normal Bivariate Density Function to solve this and related problems.

the compound call *and* its underlying call will be exercised.

The Roll-Geske-Whaley Compound Option Formula

Since American options need to allow for early exercise of the option on dividend-paying stocks, they are significantly more difficult to price than their European counterparts. Their early exercise potential creates an inequality in the Black-Scholes differential equation, which makes it considerably more difficult to solve. In fact, even the standard put-call parity condition breaks down for American options.

Black's Pseudo-American call model assumes ex-ante that the call on a dividend paying stock may be exercised early, but it does not provide a probability for this early exercise. In effect, the option holder is assumed to decide at time 0 whether he prefers to exercise the call on the ex-dividend date to receive the stock and the dividend or wait until the call's expiry and consider exercising to receive the stock without the dividend. While this assumption is not true, the option value does reflect the assumption. Roll's (also attributed to Geske and Whaley) formula corrects for this early decision assumption, but is somewhat more complex.

The Roll-Geske-Whaley model (See Roll [1977], Geske [1979a and 1979b] and Whaley [1981 and 1982]) requires that we first we find that minimum stock price (S_{tD}^*) at the ex-dividend date (t_D) that will lead to early exercise of the option. That is, on the ex-dividend date, given a known dividend payment D, there is some minimum time t_D stock price S_{tD}^* at which early exercise (on the ex-dividend date) is optimal. At this minimum price, or at a higher price, paying the exercise price and receiving the stock along with the dividend is worth more than retaining the call. Thus, it makes sense to exercise the call. All values of S_{tD} exceeding this price S_{tD}^* will lead to early option exercise. This minimum price is found by solving the following for S_{tD}^*:

(26) $$c_{tD}\left(S_{tD}^*, t_D; r, T, X, \sigma\right) = S_{tD}^* + D - X$$

When dividends are zero, the call will never be exercised early. If dividends are small, the underlying stock price S_{tD}^* at time t_D will need to be very large to justify early exercise of the call. Thus, for any dividend amount $D > 0$, there is some minimum stock price S_{tD}^* on the ex-dividend date that would lead to early exercise of the American call. There are several numerical and iterative techniques for solving this equality for S_{tD}^* (e.g., the methods of bisection and Newton Raphson), but simple substitution and iteration will work. Second, to find the call's current value c_0, solve the following:

(27) $$c_o = \left(S_o - D\cdot e^{-r\cdot t_D}\right)N(y_1) + \left(S_o - D\cdot e^{-r\cdot t_D}\right)M\left(d_1, -y_1; -\sqrt{\frac{t_D}{T}}\right)$$
$$- Xe^{-r\cdot T}M\left(d_2, -y_2; -\sqrt{\frac{t_D}{T}}\right) - \left(X - D\right)e^{-r\cdot t_D}N(y_2),$$

where

(28) $$d_1 = \frac{\left[\ln\left(\frac{S_o - D\cdot e^{-r\cdot t_D}}{X}\right) + \left(r + \frac{1}{2}\cdot\sigma^2\right)T\right]}{\sigma\sqrt{T}}$$

(29) $$d_2 = d_1 - \sigma \cdot \sqrt{T}$$

(30) $$y_1 = \frac{\left[\ln\left(\frac{S_o - D \cdot e^{-r \cdot t_D}}{S_{tD}^*} \right) + \left(r + \frac{1}{2} \cdot \sigma^2 \right) t_D \right]}{\sigma \sqrt{t_D}}$$

(31) $$y_2 = y_1 - \sigma \sqrt{t_D}$$

$M\left(d(*), y(*); \sqrt{\frac{t_D}{T}} \right)$ is a cumulative multivariate (bivariate) normal distribution, where $\sqrt{\frac{t_D}{T}}$ is the correlation coefficient between random variables S_{tD} and S_T, assuming that stock returns are independent over non-overlapping periods of time. The process of computing this cumulative probability is described in Section 2.8.

The intuition behind the Roll-Geske-Whaley model is fairly straightforward. An American call with a single ex-dividend date is essentially a compound option that can be replicated with the following portfolio:

a. A long position in a European call with expiration T and exercise price X.
b. A short position in a European compound call (call on the call in part a) with expiration t_D, and exercise price $S_{tD}^* + D - X$
c. A long position in a European call with expiration t_D and exercise price S_{tD}^*

This implies that the American call holder can hold his call to expiration as he would a European call unless it gets called away from him at time t_D by his exercising a second call at time t_D to take the underlying stock at price S_{tD}^*. The cash flows produced from this replicating portfolio on the ex-dividend date are given in Table 6. We see in Table 1 that if the dividend amount is sufficiently high such that $S_{tD} > S_{tD}^*$, the American call is exercised on the ex-dividend date. Notice that Table 1 demonstrates that the payoff structure of the portfolio exactly duplicates the required payoff structure for an American call on the date t_D, thus demonstrating that the portfolio replicates the American call.

POSITION	**$S_{tD} < S_{tD}^*$ (c Held)**	**$S_{tD} > S_{tD}^*$ (c Exercised)**
European Call (a)	$c(S_{tD}, t_D, r, \sigma, T, X)$	0 (Called away by Call (b))
Compound Call (b)	0	$S_{tD}^* + D - X$ (Exercise Money)
European Call (c)	0	$S_{tD} - S_{tD}^*$ (Call is Exercised)
Total	$c(S_{tD}, t_D, r, \sigma, T, X)$	$S_{tD} + D - X$

Table 1: Compound Call Option Payoffs on Ex-Dividend Date t_D

Illustration: Calculating the Value of an American Call on Dividend-Paying Stock

Suppose that we wish to calculate the value of a 6-month X = $45 American call on a share of stock that is currently selling for $50. The stock is expected to go ex-dividend on a $5 payment in three months, and not again until after the option expires. The standard deviation of returns on the underlying stock is .4 and the riskless return rate is 3%. What is the call worth today assuming:

a. the European Known Dividend model?
b. Black's Pseudo-American Call model?
c. the Roll-Geske-Whaley model?

a. Under the European Known Dividend model, the value of the call is $5.39, calculated as follows:

$$c_0 = (50 - 5e^{.03\times.25}) \times .5782 - \frac{45}{e^{.03\times.5}} \times .4660 = 5.39,$$

where

$$d_1 = \frac{\ln\left(\frac{50 - 5e^{.03\times.25}}{45}\right) + \left(.03 + \frac{1}{2}\times.16\right)\times.5}{.4\times\sqrt{.5}} = .1974; N(d_1) = .5782$$

$$d_2 = .5782 - .4\times\sqrt{.5} = -.0855; N(d_2) = .4660$$

b. To use Black's Pseudo-American call formula, we will first determine whether dividend exceeds the ex-dividend date time value associated with exercise money: 5 > $45[1-e^{-.03(.5-.25)}]$ = .336. Since the dividend does exceed this value, we will calculate the value of the call assuming that we exercise it just before the underlying stock goes ex-dividend in 3 months, while it trades with dividend:

$$c_0 = 50 \times .7468 - \frac{45}{e^{.03\times.25}} \times .6788 = 7.02,$$

where

$$d_1 = \frac{\ln\left(\frac{50}{45}\right) + \left(.03 + \frac{1}{2}\times.16\right)\times.25}{.4\times\sqrt{.25}} = .6643; N(d_1) = .7468$$

$$d_2 = .6643 - .4\times\sqrt{.25} = .4643; N(d_2) = .6788$$

Under Black's Pseudo-American call formula, we find that the value of the call is *MAX*[5.39, 7.02], or 7.02.

c. Using the Geske-Roll-Whaley Model, we first calculate S_{tD}^*, the critical stock value on the ex-dividend date required for early exercise of the call:

$$c_{tD}^* = S_{tD}^* + D - X$$

$$c_{tD}(S_{tD}^*, .25, .03, .5, 45, .4) = S_{tD}^* + 5 - 45.$$

Using simple substitution or an appropriate numerical or iterative technique, one finds that:

$$c_{tD}(42.4924, .25, .03, .5, 45, .4) = 2.4924 = 42.4924 + 5 - 45$$

$$S_{tD}^{*} = 42.4924$$

This means that if the stock is selling for $S_{tD}{}^{*}$ = \$42.4924 in 3 months (t_D = .25), the call on the stock trading ex-dividend will have the same value as the stock, plus declared dividend minus the exercise money. Thus, as long as the ex-dividend value of the stock plus the dividend minus exercise money exceeds the ex-dividend call value, the call will be exercised early. The value of the American call is found to be 6.96, calculated from equation (27) as follows:

$$c_o = \left(50 - 5\cdot e^{-.03\times.25}\right)\cdot .6658 + \left(50 - 5\cdot e^{-.03\times.25}\right)\cdot .0798$$
$$- 45\cdot e^{-.03\times.5}\cdot .07175 - \left(45-5\right)\cdot e^{-.03\times.25}\cdot .5903 = 6.96$$

where

$$d_1 = \frac{\left[\ln\left(\frac{50-5\cdot e^{-.03\times.25}}{45}\right) + \left(.03 + \frac{1}{2}\cdot .16\right)\cdot .5\right]}{.4\cdot\sqrt{.5}} = .1974$$

$$d_2 = .1974 - .4\cdot\sqrt{.5} = -.0855$$

$$y_1 = \frac{\left[\ln\left(\frac{50-5\cdot e^{-.03\times.25}}{42.4924}\right) + \left(.03 + \frac{1}{2}\cdot .16\right)\cdot .25\right]}{.4\cdot\sqrt{.25}} = .4283$$

$$y_2 = .4283 - .4\cdot\sqrt{.25} = .2283$$

The Roll-Geske-Whaley model provided the correct price of \$6.96 for the American call. Black’s approximation (\$7.02) provided a reasonable estimate, while the European Known Dividend model, as expected, undervalued the call (\$5.39) relative to Black's model.

Put-Call Parity for Compound Options

Here, using the notation from above, we set forth pricing formulas for other compound options, including a call on a call (repeated from above), call on a put, put on a call and put on a put:

$$c_{0,call} = S_0 M\left(d_1, y_1; \sqrt{\frac{T_1}{T_2}}\right) - X_2 e^{-rT_2} M\left(d_2, y_2; \sqrt{\frac{T_1}{T_2}}\right) - X_1 e^{-rT_1} N(d_2)$$

$$p_{0,call} = X_2 e^{-rT_2} M\left(-d_2, y_2; -\sqrt{\frac{T_1}{T_2}}\right) - S_0 M\left(-d_1, y_1; -\sqrt{\frac{T_1}{T_2}}\right) + X_1 e^{-rT_1} N(-d_2)$$

$$c_{0,put} = X_2 e^{-rT_2} M\left(-d_2, -y_2; \sqrt{\frac{T_1}{T_2}}\right) - S_0 M\left(-d_2, -y_2; \sqrt{\frac{T_1}{T_2}}\right) - X_1 e^{-rT_1} N(-d_2)$$

$$p_{0,put} = S_0 M\left(d_1, -y_1; -\sqrt{\frac{T_1}{T_2}}\right) - X_2 e^{-rT_2} M\left(d_2, -y_2; -\sqrt{\frac{T_1}{T_2}}\right) + X_1 e^{-rT_1} N(d_2)$$

Let $c_{t,call}$, $p_{t,call}$, $c_{t,put}$, and $p_{t,put}$ denote the values of a call on a call, put on a call, call on a put, and put on a put at time t, respectively. To obtain the formulas from our compound call pricing formula, first recall our compound option payoff functions from Section 7.6.1. We can demonstrate that the above pricing functions for our other compound options hold by first working with the exercise date T_1 payoff function for a portfolio, say, with a long position of one call on a call and a short position of one put on a call. At expiration date T_1, we have for our portfolio:

$$c_{T1,call} - p_{T1,call} = \text{MAX}[c_{u,T1} - X_1, 0] - \text{MAX}[X_1 - c_{u,T1}, 0] = c_{u,T1} - X_1.$$

For any earlier time $t < T_1$, the value of the portfolio with time T_1 cash flow $c_{u,T1} - X_1$ must equal $c_{u,t} - X_1 e^{-r(T1-t)}$ since we must discount the cash associated with the exercise money X. In particular, at time 0, the value of the portfolio is:

$$c_{0,call} - p_{0,call} = c_{u,0} - X_1 e^{-rT_1}.$$

This formula is the put-call parity relation for the options on the underlying call. Note that $c_{u,0}$ is the present value of the underlying call and $X_1 e^{-rT1}$ is the present value of the exercise money associated with the compound call.

Next, we demonstrate that the two relevant time zero pricing formulas for the call on the call and the put on the call above are consistent with this portfolio value. We will demonstrate that our call on call value minus our put on call value yields the same value as our put-call formula for the options on the underlying call above:

$$S_0 M\left(d_1, y_1; \sqrt{\frac{T_1}{T_2}}\right) - X_2 e^{-rT_2} M\left(d_2, y_2; \sqrt{\frac{T_1}{T_2}}\right) - X_1 e^{-rT_1} N(d_2)$$
$$- \left[X_2 e^{-rT_2} M\left(-d_2, y_2; -\sqrt{\frac{T_1}{T_2}}\right) - S_0 M\left(-d_1, y_1; -\sqrt{\frac{T_1}{T_2}}\right) + X_1 e^{-rT_1} N(-d_2)\right] = c_{u,0} - X_1 e^{-rT_1}$$

Because the normal curve is symmetric, implying that the total area under the normal curve is $N(d_2) + N(-d_2) = 1$, this simplifies as follows:

$$S_0 M\left(d_1, y_1; \sqrt{\frac{T_1}{T_2}}\right) - X_2 e^{-rT_2} M\left(d_2, y_2; \sqrt{\frac{T_1}{T_2}}\right) - X_1 e^{-rT_1} - \left[X_2 e^{-rT_2} M\left(-d_2, y_2; -\sqrt{\frac{T_1}{T_2}}\right) - S_0 M\left(-d_1, y_1; -\sqrt{\frac{T_1}{T_2}}\right)\right] = c_{\mathrm{u},0} - X_1 e^{-rT_1},$$

and with some minor rearrangement, is rewritten:

$$S_0\left[M\left(d_1, y_1; \sqrt{\frac{T_1}{T_2}}\right) + M\left(-d_1, y_1; -\sqrt{\frac{T_1}{T_2}}\right)\right] - X_2 e^{-rT_2}\left[M\left(d_2, y_2; \sqrt{\frac{T_1}{T_2}}\right) + M\left(-d_2, y_2; -\sqrt{\frac{T_1}{T_2}}\right)\right] - X_1 e^{-rT_1} = c_{u,0} - X_1 e^{-rT_1}.$$

Because $N(y_1) = M\left(d_1, y_1; \sqrt{\frac{T_1}{T_2}}\right) + M\left(-d_1, y_1; -\sqrt{\frac{T_1}{T_2}}\right)$, and $N(y_2) = M\left(d_2, y_2; \sqrt{\frac{T_1}{T_2}}\right) + M\left(-d_2, y_2; -\sqrt{\frac{T_1}{T_2}}\right)$, we have:[3]

$$S_0 N(y_1) - X_2 e^{-rT_2} N(y_2) - X_1 e^{-rT_1} = c_{u,0} - X_1 e^{-rT_1}.$$

$$S_0 N(y_1) - X_2 e^{-rT_2} N(y_2) = c_{u,0}$$

The final equality above is true because it is consistent with the European call value as derived earlier. This demonstrates that our put on call formula is consistent with our call on call formula, which was reasoned earlier. Similar manipulations can be used to verify our other compound option formulas and our compound option put-call parity formula depicted in Table 2. This simplifies to our standard plain vanilla put-call parity relation.

Present Value:

$$c_{0,call} \quad - \quad p_{0,call} \quad = \quad c_{0,put} \quad - \quad p_{0,put} \quad + S_0 - X_2 e^{-rT_2}$$

Time T_1 Value:

MAX[$c_{u,T1}$ -X_1,0] - MAX[X_1 - $c_{u,T1}$,0] = MAX[$p_{u,T1}$ -X_1,0]-MAX[X_1 - $p_{u,T1}$,0] $+ S_0 - X_2 e^{-rT_2}$

$c_{u,T1}$ -X_1 = $p_{u,T1}$ -X_1 $+ S_0 - X_2 e^{-rT_2}$

Table 2: Put Call Parity Relation for Compound Options

[3] It is fairly straightforward to verify these relationships by using a multinomial normal distribution calculator and a z-table.

B. **Changing the Pricing Numeraire**

Using our notation from Chapter 7, the exchange option is valued in risk-neutral probability space as:

$$E_{\mathbb{Q}}[V_T|\mathcal{F}_0] = E_{\mathbb{Q}}[c_T|\mathcal{F}_0] = E_{\mathbb{Q}}\left[MAX\left[S_{2,T} - S_{1,T}, 0\right]\middle|\mathcal{F}_0\right]$$

However, we can change our numeraire to stock 1. Now, the exchange option can be valued in a Black-Scholes environment with stock 1 as the numeraire:

$$\frac{c_0}{S_{1,0}} = E_{\mathbb{Q}}\left[MAX\left[\frac{S_{2,T}}{S_{1,T}} - 1,0\right]\middle|\mathcal{F}_0\right]$$

$$\frac{c_0}{S_{1,0}} = \frac{S_{2,0}}{S_{1,0}} N(d_1) - N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_{2,0}}{S_{1,0}}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \ln\left(\frac{S_{2,0}}{S_{1,0}}\right) - \frac{1}{2}\sigma\sqrt{T}$$

Thus, ignoring dividends, we calculate the value of a single exchange call (c_0) as follows:

$$c_0 = S_{2,0}N(d_1) - S_{1,0}N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_{2,0}}{S_{1,0}}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \ln\left(\frac{S_{2,0}}{S_{1,0}}\right) - \frac{1}{2}\sigma\sqrt{T}$$

It is interesting to note that not even the riskless return rate plays a role in valuing the exchange option in the absence of dividends; in effect, the riskless return is embedded in the price of stock 1 in the risk neutral environment. Recall that we used fairly minor variations of this equation in Chapters 7 and 9 to value plain vanilla calls (Just assume that asset 1 is riskless with value X). Following the procedure set forth in Section 7.D leads to the solution for c_0 given above for the exchange option.

C. Exchange Options

<u>The Margrabe Model</u>

In this section, we discuss a variation of the Black-Scholes Model provided by Margrabe [1978] for the valuation of an option to exchange one risky asset for another. Suppose that the prices S_1 and S_2 of two assets follow geometric Brownian motion processes with σ_1 and σ_2 as standard deviations of logarithms of price relatives (or returns) for each of the two securities:

$$\frac{dS_1}{S_1} = \mu_1 dt + \sigma_1 dZ_1$$

$$\frac{dS_2}{S_2} = \mu_2 dt + \sigma_2 dZ_2$$

where $E[dZ_1 \times dZ_2] = \rho_{1,2}dt$, where $\rho_{1,2}$ is the correlation coefficient between logarithms of price relatives $ln(S_1/S_2)$ between the two securities. Furthermore, the variance of logarithms of price relatives of the two assets relative to one another is σ^2:[4]

$$\sigma^2 = \sigma^2_{ln\left(S_1/S_2\right)} = \sigma^2_{ln(S_1)} + \sigma^2_{ln(S_2)} - 2\rho_{1,2}\sigma_{ln(S_1)}\sigma_{ln(S_2)}$$

That is, σ is the anticipated standard deviation of $[ln(S_1/S_2)]$ over the life of the option. We will use the standard deviation of logs of relative returns σ^2 for our option valuation. Note that asset 1 is potentially given up in the exchange and asset 2 is potentially received.

The Garman- Köhlagen Model

In this section, we discuss a variation of the Black-Scholes Model provided by Garman and Köhlagen [1983], Biger and Hull [1983] and Grabbe [1983] for the valuation of currency options. This option is much like the exchange option presented above. The stock option pricing model is based on the assumption that investors price calls as though they expect the underlying stock to earn the risk-free rate of return. However, when we value currency options, we acknowledge that interest rates may differ between the foreign and domestic countries. Hence, we quote two interest rates $r(f)$ and $r(d)$, one each for the foreign and domestic currencies. The standard differential equation for currency options is as follows:

$$\frac{\partial c}{\partial t} = r(d)c - [r(d) - r(f)]\frac{\partial c}{\partial s}s - \frac{1}{2}s^2\sigma^2\frac{\partial^2 c}{\partial s^2}$$

where s is the exchange rate. Notice that this differential equation is identical to the Merton dividend leakage model above, where the dividend yield δ is replaced by the foreign interest rate $r(f)$ and $r(d)$ is simply the domestic riskless rate. Its particular solution, subject to the *boundary condition* $c_T = MAX[0, s_T - X]$, is given as follows:

$$c_0 = \frac{s_0}{e^{r(f)T}}N(d_1) - \frac{X}{e^{r(d)T}}N(d_2) = c_0[s_0, T, r(f), r(d)\sigma, X]$$

$$d_1 = \frac{\ln\left(\frac{s_0}{X}\right) + \left(r(d) - r(f) + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

[4] End of chapter Exercise 7.16 provides a verification of this equality.

$$d_2 = d_1 - \sigma\sqrt{T}$$

where $r(d)$ is the domestic riskless rate (the rate for the currency that will be given up if the option is exercised), $r(f)$ is the foreign riskless rate (the rate for the currency which may be purchased). The spot rate for the currency is s_0. The standard deviation of proportional changes in the currency value to be purchased in terms of the domestic currency is σ. The exercise price of the option, X, represents the number of units of the domestic currency to be given up for the foreign currency.

Currency Option Illustrations

Consider the following example where call options are traded on Brazilian real (BRL). One U.S. dollar is currently worth 2 Brazilian real ($s_0 = .5$). We wish to evaluate a 2-year European call and put on BRL with exercise prices equal to $X = .45$. U.S. and Brazilian interest rates are .03 and .08, respectively. The annual standard deviation of exchange rates is .15. We calculate $d_1 = .1313$ and $d_2 = -.0808$; $N(d_1)$ and $N(d_2)$ are .5522 and .4678, respectively. Thus, the call is valued at $c_0 = .037$ and the put is valued at $p_0 = .035$.

Exchange Option Illustration

Suppose that the Predator Company has announced its intent to acquire the Prey Company through an exchange offer. That is, the Predator Company has extended an offer to Prey Company shareholders to exchange one share of its own stock for each share of Prey Company stock. This offer expires in 90 days (.25 years). Shares of stock in the two companies follow geometric Brownian motion processes with a variance of $\sigma^2_{\ln(S1)} = .16$ for Predator (Predator is company 1) and $\sigma^2_{\ln(S2)} = .36$ for Prey, and zero correlation $\rho_{1,2} = 0$ between the two. Predator Company stock is currently selling for \$40 and Prey shares are selling for \$50. The current riskless return rate is .05. What is the value of the exchange option associated with this tender offer? First, we calculate σ^2, the variance of changes (actually, logs thereof) of stock prices relative to each other as follows:

$$\sigma^2 = \sigma^2_{ln\left(S_1/S_2\right)} = \sigma^2_{ln(S_1)} + \sigma^2_{ln(S_2)} - 2\rho_{1,2}\sigma_{ln(S_1)}\sigma_{ln(S_2)} = .16 + .36 - 2 \times 0 \times .4 \times .6 = .52$$

The value of this exchange option is calculated as 12.61 as follows:

$$c_0 = 50N(d_1) - 40N(d_2) = 50 \times .7879 - 40 \times .6695 = 12.61$$

$$d_1 = \frac{\ln\left(\frac{50}{40}\right) + \left(\frac{1}{2} \times .7211^2\right) \times .25}{.7211 \times \sqrt{.25}} = .79917;\ \ N(d_1) = .7879$$

$$d_2 = .79917 - .7211\sqrt{.25} = .4386;\ \ N(d_2) = .6695$$

With put-call parity, which is unchanged from the usual plain vanilla scenario, we find that the put is worth 2.61 in the Black-Scholes environment. Note that a long position in one call plus a short position in one put has a combined value equal to 10, the difference between the prices of the two shares. A long position in the exchange call and a short position in the exchange put essentially means a one-for-one exchange of shares produces a profit of 10 to the portfolio holder.

Creating a call option to "buy" one share of Predator Company with one share of Prey Company is identical to creating a put option to "sell" one share of Prey Company for one share of Predator Company. Working through the Black-Scholes equations above, reversing the prices of stock 1 and stock 2, we find that the exchange put is worth 12.61, just as was the call price earlier. The exchange call in this reversed scenario is worth 2.61.

D. Hedging Exchange Exposure with Currency Options

Multinational companies face a number of risks not experienced by companies operating in only one country. Among the most significant of these risks is exchange rate uncertainty and fluctuations. Based on the example introduced in Chapter 5, suppose that the Dayton Company of America expects to receive a payoff of £1,000,000 in three months, and intends to convert its cash flows to dollars. Continue to assume relevant data as follows:

Spot exchange rate: $1.7640/£
Three-month forward exchange rate: $1.7540/£
U.K. Borrowing interest rate: 10.0%
U.S. Borrowing interest rate: 8.0%
U.K. Lending interest rate: 8.0%
U.S. Lending interest rate: 6.0 %

Also assume that there exist call and put options and forward contracts with the following terms, and that their premiums might not reflect formula values:

Term to options expiration: 3 months
Exercise price: $1.75 per £
Put Premium: $0.025 per £
Call Premium: $.065 per £
Brokerage cost per options contract on £31,250: $50

Our problem is to evaluate methods of managing the transaction risk associated with this extension of credit and the implications of each.

We will consider two options-based hedging strategies here. The first is the put hedge (partial hedge) strategy which involves the purchase of a put on pounds, enabling the firm to protect itself against devaluation of pounds. If the value of pounds increases, the firm realizes a greater profit. However, the firm must pay the full cost of the put. With the conversion or call and put hedge strategy, the proceeds from the sale of a call are used to offset the purchase price of the put. This strategy acts as a collar, locking in the value of pounds at the originations of the options contracts. Hence, the firm does not benefit from any appreciation in the value of the pound.

First, we consider the Put Hedge strategy. We will purchase three month put options on £1,000,000 with an exercise price of $1.75/£ with a total premium of $25,000. Time zero brokerage costs total $1,600 (32 contracts at $50 per contract). Thus, the total time zero cash outlay is $26,600. Forgone interest on the sum of the premium and brokerage costs totals $399.

Expressed in terms of future value, the total cash outlay is $26,999. The result of this strategy is that the firm receives one of the following in three months:

1. An unlimited maximum less the $26,999 premium and brokerage fees. The dollar value of this strategy increases as the value of the dollar drops against the pound. Since cash flows are not certain, this hedge is considered partial.
2. A minimum of $1,750,000 less $26,999 for a net of $1,723,001. This minimum value to be received may be unacceptably low; however, there is upside cash flow potential.

Alternatively, the firm can employ the conversion or the Call and Put Hedge. This strategy involves the combination of calls and puts, such that total risk can be eliminated. Consider the writing of a call with an exercise price of $1.75 expiring in three months along with the purchase of a put with the same terms. The time zero cash flows are summarized as follows:

Put Premium.......	- $25,000	Call Premium.......	+ $65,000
Put brokerage fee.	- $ 1,600	Call brokerage fee.	- $ 1,600
	Net Time zero cash flows		+ $36,800

The result of this conversion is that the interest earned on the net time zero outlay is $552. If the three-month exchange rate is less than $1.75/£, the exchange rate of $1.75/£ is locked in by the put. If the exchange rate exceeds $1.75/£, the obligation incurred by the short position in the call is activated. Thus, the firm's exchange rate of $1.75/£ is locked in no matter what the market exchange rate is. The cash flows in three months are summarized as follows:

Put cash flows (£1,000,000 × MAX[1.75-S_1,0])
Call cash flows (£1,000,000 × MIN[1.75-S_1,0])
Total of option transactions:

£1,000,000 × (1.75 - S_1)	= $1,750,000 - (£1,000,000 × S_1)
Exchange of Currency	= (£1,000,000 × S_1)
Time zero cash flows	= $ 36,800
Interest on Time zero flows	= $ 552
TOTAL TIME ONE CASH FLOWS	= $1,787,352

This cash flow of $1,787,352 is assured in the absence of default risk.

E. Exotic Options

As described earlier, options confer the right but not obligation to buy or sell an underlying asset at a striking price on or before the option expiration date. We discussed earlier the "plain vanilla" options with the most simple terms. A variety of other options, generally known collectively as exotic options, exist to fulfill a variety of financial needs. In this chapter, we will discuss a small sampling of the many classes of categories of exotic options and a sampling of option types in these classes.

Locking in Profits

As we will discuss in greater detail in the next chapter, *collar* is a combination of options designed to maintain the investor's profit (or loss) in an underlying asset within a specified range. For example, a long position in a put will protect the investor in an underlying asset from price decreases. The cost of this long position in the put can be offset by the investor selling a call on the same asset, thereby giving up some of his profits should the underlying asset value rise. Hence, the investor, by buying a put and selling a call, locks in his profits within a given range. A z*ero-cost collar* is a package of options (for example, a long call and short put) designed to require zero net investment. Typically, the exercise price of the put is set at a level relative to the exercise price of the call so that their values exactly offset each other.

Similarly, a long position in a *Range Forward Contract* enables (and obliges) its owner to purchase the underlying security at time T value for the following price: (X_1 if $S_T \geq X_1$; S_T if $X_1 > S_T \geq X_2$; or X_2 if $S_T < X_2$). This position replicates a short position in a collar. If the objective were to replicate a long position in a zero-cost collar, the two exercise prices would be set so that no initial net investment is required to enter into the contract. These collars are very useful for investors who wish to lock in profit levels of their investments without selling their securities.

Path Dependent Options

A path dependent option's payoff is a function of the path that the underlying asset takes prior to option exercise. Thus, the exercise price of the option is not a simple constant as in the case of a plain vanilla option.

An *Asian Option* (average rate) is based on the average price (or exchange rate) of the underlying asset (or currency). For example, an Asian call on currency permits its owner to receive the difference between the average underlying asset price over the life of the option (A_T) and the exercise price (X) associated with the option: $C_{A,T} = MAX(0, A_T - X)$ where $A_T = \sum S_t/n$. A potential user of an Asian option might be an exporter who sells her production to a particular country the same number of units of its product each day. Since the exchange rate will vary daily, the revenues received by the exporter will vary. The Asian option enables the exporter to stabilize her cash flows without entering the derivatives market on a daily basis. The cash flow structures of these options vary from contract to contract. For example, some contracts call for the payoff to be related to the difference between the time T spot rate and the average exchange rate realized during the life of the option.

A *lookback option* enables its owner to purchase (or sell in the case of a put) the underlying security at the lowest price (or highest price in the case of a put) realized over the life of the option. Such options might be very desirable for an investor who wishes to take a position in the underlying asset but lacks the ability to time this transaction so that he buys at the lowest price or sells at the highest realized price over a given period of time.

A *barrier option* is similar to a "plain vanilla" option except that it expires (in the case of a down-and-out option) or is activated (in the case of up-and-in option) once the underlying asset value reaches a pre-specified price. These are often referred to as either *knock out* or *knock in* options.

Other Exotic Options

A *chooser option* provides its owner to choose at time $t_1 < T$ between converting the chooser option to either a plain vanilla call or put that expires at time T. At the time of choosing t_1, the chooser option is valued as:

$$c_{c,t_1} = MAX[c_{t_1}, p_{t_1}] = MAX[c_{t_1}, c_{t_1} + Xe^{-r_f T - t_1} - S_{t_1}] = c_{t_1} + MAX[0, Xe^{-r_f (T-t_1)} - S_{t_1}]$$

Thus, the time zero (or time t_1) value of a chooser option with choice date t_1 and expiration date T is equivalent to a portfolio made up of a call option with exercise price *X* that expires at time *T* and a put with exercise price $Xe^{-rf(T-t1)}$.

Rainbow options are written on two or more assets. A rainbow call may give its owner the right to choose between any of two or more assets.

A *digital option* pays either 1 or 0, depending on the occurrence of a particular event. For example, a digital option might pay 1 if and only if a terminal stock price exceeds an exercise price, and pay zero otherwise.

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Exercises

1. Consider a compound call on a call with exercise price $X_1 = 5$, expiring in six months ($T_1 = .5$) on an equity call, with exercise price $X_2 = 75$, expiring in one year ($T_2 = 1$). The stock currently sells for $S_0 = 70$ and has a return volatility equal to $\sigma = .4$. The riskless return rate equals $r = .05$. What is the value of the compound call?

2. Suppose that S_1 and S_2 are random variables. In our exchange option discussion, we claimed that we could measure the variance of logarithmic of price relatives of the two assets relative to one another with a particular variant of σ^2:

$$\sigma^2_{ln\left(S_1/S_2\right)} = \sigma^2_{ln(S_1)} + \sigma^2_{ln(S_2)} - \rho_{1,2}\sigma_{ln(S_1)}\sigma_{ln(S_2)}$$

a. Demonstrate more generally that $Var[S_1 - S_2] = Var[S_1] + Var[S_2] - Cov[S_1, S_2]$.

b. Based on your demonstration for part a, demonstrate the following:

$$\sigma^2_{ln\left(S_1/S_2\right)} = \sigma^2_{ln(S_1)} + \sigma^2_{ln(S_2)} - 2\rho_{1,2}\sigma_{ln(S_1)}\sigma_{ln(S_2)}$$

3. Six-month currency options on Thai Baht (THB) denominated in U.S. dollars with exercise prices equal $0.03. The U.S. interest rate is .03 and the Thai rate is .10. The current exchange rate is $0.032/THB and the standard deviation associated with the exchange rate is .01. What is the value of this FX call? What is the value of a put with the same exercise terms?

4. A currency option series on Australian dollars (AUD) denominated in U.S. dollars. Suppose that a 2-year call option is available on AUD with an exercise price equal to $0.65. The U.S. interest rate is .04 and the Australian rate is .06. The current exchange rate is $0.7/AUD and the standard deviation associated with the exchange rate is .3. What is the value of this currency call? What is the value of a put with the same exercise terms?

5. The Smedley Company sold products to a Japanese client for ¥15,000,000. Payment is due six months later. Relevant data is as follows:

Spot exchange rate: ¥105/$
Japanese Borrowing interest rate: 9.0%
U.S. Borrowing interest rate: 7.0%
Japanese Lending interest rate: 7.0%
U.S. Lending interest rate: 5.0 %

There exist currency call and put options with the following terms:

Size of options contracts: ¥1,000,000
Term to expiration of options contracts: 6 months
Exercise price of put and call: $.009/¥
Put Premium: $0.00001/¥
Call Premium:$.0001/¥
Brokerage cost per options contract: $50

Discuss the implications associated with each of the options-based methods for managing the transactions exposure risk associated with this extension of credit.

6. A *chooser option* (or, *as you like it option*) is sometimes held to allow its owner to benefit from underlying security volatility, as in 1991, when the Persian Gulf War led to high levels of

price volatility in oil markets. The *standard chooser option* allows its owner at a *choice date* $0 < t_c < T$ to choose between the option taking the form of a plain vanilla call or a put. Thus, on the choice date t_c, the owner of the standard chooser option decides if he wishes the option to be a plain vanilla call or a put for the remainder of its life.

a. Write a function that gives the choice date t_c payoff for the standard chooser option in terms of the plain vanilla options from which the owner can choose.

b. Making use of the put-call relation for plain vanilla options, evaluate as of time t_c the plain vanilla put component of the chooser option.

c. Making use of your solution to part b, rewrite the function from part a that gives the choice date t_c payoff for the standard chooser option in terms of the plain vanilla options from which the owner can choose. Comment on your findings.

d. Under what circumstances does the owner of the chooser option select the call on the choice date t_c? Under what circumstances does the owner of the chooser option select the put on the choice date t_c?

e. Devise a model to evaluate the chooser option in a Black-Scholes environment.

f. An investor needs to evaluate a one-year chooser option on a non-dividend-paying stock in a Black-Scholes environment. The choice date is in 6 months. The riskless return rate is .05 and the standard deviation of underlying stock returns is .4. Both the current stock price and the exercise price of the chooser option are 50. What is the current value of the chooser option?

Solutions

1. First, we calculate the critical value S^*_{T1} for underlying option exercise at time T_1. A search process reveals that this equals 66.578906:

$$c_{u,T1} = S^*_{T1} \times N\left(\frac{\left[ln\left(\frac{S^*_{T1}}{75}\right) + \left(.05 + \frac{1}{2} \times .16\right) \times (1 - .5)\right]}{.4\sqrt{.5 - .25}}\right)$$
$$- \frac{75}{e^{.05 \times (1-.5)}} N\left(\frac{\left[ln\left(\frac{S^*_{T1}}{75}\right) + \left(.05 - \frac{1}{2} \times .16\right) \times (1 - .5)\right]}{.4 \times \sqrt{1 - .5}}\right) = X_1 = 5;\ S^*_{T1}$$
$$= 66.578906$$

Note that the correlation coefficient between our two random variables is $\rho = \sqrt{.5/1} = .7071$. We use the reasoning from Section 5.4 and the bivariate normal distribution to calculate the compound call value as follows:

$$c_0 = 70 \times .4834 - 75 \times e^{-(.05 \times 1)} \times .3382 - 5e^{-(.05 \times .5)} \times .5494 = 7.03314$$
$$d_1 = \frac{\left[ln\left(\frac{70}{66.578906}\right) + \left(.05 + \frac{1}{2} \times .16\right) \times .5\right]}{.4 \times \sqrt{.5}} = .4070$$
$$d_2 = .4070 - .4 \times \sqrt{.5} = .1241$$
$$y_1 = \frac{\left[ln\left(\frac{70}{75}\right) + \left(.05 + \frac{1}{2} \times .16\right) \times 1\right]}{.4 \times \sqrt{1}} = .1525$$
$$y_2 = .1525 - .4 \times \sqrt{1} = -.2475$$

Thus, we find that the value of this compound call equals 7.03314. The bivariate probabilities were calculated using a spreadsheet-based multinomial cumulative distribution calculator.

2. Suppose that S_1 and S_2 are random variables. In our exchange option discussion in Section 7.8.1, we claimed that we could measure the variance of logarithmic of price relatives of the two assets relative to one another with a particular variant of σ^2:

$$\sigma^2_{ln\left(S_1/S_2\right)} = \sigma^2_{ln(S_1)} + \sigma^2_{ln(S_2)} - \rho_{1,2}\sigma_{ln(S_1)}\sigma_{ln(S_2)}$$

a. Demonstrate more generally that $Var[S_1 - S_2] = Var[S_1] + Var[S_2] - Cov[S_1,S_2]$.
b. Based on your demonstration for part a, demonstrate the following:

$$\sigma^2_{ln\left(S_1/S_2\right)} = \sigma^2_{ln(S_1)} + \sigma^2_{ln(S_2)} - 2\rho_{1,2}\sigma_{ln(S_1)}\sigma_{ln(S_2)}$$

3. To answer this question, we first calculate d_1:

$$d_1 = \frac{ln\left(\frac{.032}{.03}\right) + \left(.03 - .1 + \frac{.01^2}{2}\right) \cdot .5}{.01\sqrt{.5}} = 4.180913$$

Next we calculate d_2:

$d_2 = d_1 - \sigma\sqrt{T} = 4.180913 - .007071 = 4.173842$

Next, find cumulative normal density functions (z-values) for d_1 and d_2:

$N(d_1) = N(4.180913) = .99998548$

$N(d_2) = N(4.173842) = .99998502$

Finally, we value the call as follows:

$c_0 = .032\times e^{-.1\times.5}\times.99998548 - .03\times e^{-.03\times.5}\times.99998502 = \$.00088598$

We can evaluate the European put using put-call parity as follows:

$p_0 = c_0 + Xe^{-r(d)T} - s_0e^{-r(f)T} = .00088598 + .03e^{-.03\times.5} - .032e^{-.1\times.5} = .000000001$

The put is valued at slightly more than 0. The estimates for the cumulative normal distributions will be great enough to make it appear as though the put has negative value. We'll accept that we estimated with error and note that the put value always exceeds zero.

4. To answer this question, we first calculate d_1:

$$d_1 = \frac{\ln\left(\frac{.7e^{-.06\cdot 2}}{.65e^{-.04\cdot 2}}\right) + \left(.06 - .04 + \frac{.3^2}{2}\right)\cdot 2}{.3\sqrt{2}} = .1983$$

Next we calculate d_2:

$d_2 = d_1 - \sigma\sqrt{T} = .1983 - .4242 = -.226$

Next, find cumulative normal density functions (z-values) for d_1 and d_2:

$N(d_1) = N(.1983) = .5786$

$N(d_2) = N(-.226) = .4106$

Finally, we value the call as follows:

$c_0 = .65(.5786) - [.7\times.8869]\times(.4106) = \$0.1128$

We can evaluate a put for this European currency option series using Equation (4) as follows:

$p_0 = c_0 + Xe^{-r(d)T} - S_0e^{-r(f)T} = .1128 + .65e^{-.06\times.5} - .7e^{-.04\times.5} = .092$

5. The two relevant options market hedging strategies are the put hedge strategy and the conversion. These are described as follows:

*Put Hedge***:**

Strategy: Purchase a 6-month put option on ¥15,000,000 with an exercise price of $.009/¥ and a premium of $150. Time zero brokerage costs total $750 (15 contracts at $50 per contract - pretty high, given the premiums involved). Thus the total time zero cash outlay is $900. Expressed in terms of future value, the total cash outlay is $922.50 since interest forgone on the sum of the premium and brokerage costs totals $22.50.

Result: Receive one of the following in six months:

1. An unlimited potential maximum less the $922.50 premium and brokerage fees. The dollar value of this strategy increases as the value of the dollar drops.
2. A minimum of $135,000 less $922.50 for a net of $134,077.50. This minimum value to be received might be unacceptably low for some managers; however, there is upside cash flow potential.

Conversion or Call and Put Hedge:

Strategy: Through the combination of short call and long put positions, total risk can be eliminated. Consider the writing of a call with an exercise price of $.009 expiring in six months along with the purchase of a put with the same terms. The time zero cash flows are summarized as follows:

Put Premium........ - $ 150   Call Premium....... + $ 1,500

Put brokerage fee. - \$ 750 Call brokerage fee. - \$ 750

Net Time zero cash flows -\$150

Result: The forgone interest on the net time zero outlay is \$3.75. If the six month exchange rate is less than \$.009/¥, the exchange rate of \$.009/¥ is locked in by the put. If the exchange rate exceeds \$.009/¥, the obligation incurred by the short position in the call is activated. Thus, the exchange rate of \$.009/¥ is locked in no matter what the exchange rate is. The cash flows in six months are summarized as follows:

Put cash flows (¥15,000,000 × MAX[.009-S_1,0])

Call cash flows (¥15,000,000 × MIN[.009-S_1,0])

Total of option transactions:

(¥15,000,000 * (.009 - S_1)	=	\$135,000 - (¥15,000,000 × S_1)
Exchange of Currency	=	(¥15,000,000 × S_1)
Time zero cash flows	=	\$ -150
Interest on Time zero flows	=	\$ 3.75
TOTAL TIME ONE CASH FLOWS	=	\$134,846.25

6.a. The time t_c payoff function for the chooser option equals MAX[c_{tc}, p_{tc}].

b. By put-call parity, $p_{tc} = c_{tc} + Xe^{-r(T-tc)} - S_{tc}$

c. MAX[c_{tc}, p_{tc}] = MAX[c_{tc}, $c_{tc} + Xe^{-r(T-tc)} - S_{tc}$] = c_{tc} + MAX[0, $Xe^{-r(T-tc)} - S_{tc}$]. Pay special attention to the far right side of this equality because we will use it to value the chooser option. The far right side of the equality implies that the time t_c payoff function for a chooser option is the equivalent of a portfolio of a call on the underlying, with exercise price X and expiration date T plus a put on the underlying with exercise price equal to $Xe^{-r(T-tc)}$ and expiration date t_c.

d. The owner of the chooser option selects the call on the choice date if $c_{tc} > p_{tc}$, which will hold if $c_{tc} > c_{tc} + Xe^{-r(T-tc)} - S_{tc}$, or if $Xe^{-r(T-tc)} < S_{tc}$. Otherwise, he selects the put.

e. Based largely on our solution to part c above (in particular, the far right term), we value the chooser option as follows:

$$V_{chooser} = \left[S_0N(d_1) - \frac{X}{e^{rT}}N(d_2)\right] + \left[Xe^{-r(T-t_c)}\,N(-d_4) - S_0N(-d_3)\right]$$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$d_3 = \frac{\ln\left(\frac{S_0}{Xe^{-r(T-t_c)}}\right) + \left(r + \frac{1}{2}\sigma^2\right)t_c}{\sigma\sqrt{t_c}} = \frac{\ln\left(\frac{S_0}{Xe^{r(-T+t_c)}}\right) + rt_c + \frac{1}{2}\sigma^2 t_c}{\sigma\sqrt{t_c}}$$

$$= \frac{\ln\left(\frac{S_0}{X}\right) - \ln\left(e^{r(-T+t_c)}\right) + rt_c + \frac{1}{2}\sigma^2 t_c}{\sigma\sqrt{t_c}} = \frac{\ln\left(\frac{S_0}{X}\right) + rT + \frac{1}{2}\sigma^2 t_c}{\sigma\sqrt{t_c}}$$

$$d_4 = d_3 - \sigma\sqrt{t_c}$$

f. Value the chooser option as follows:

$$V_{chooser} = 50 \times N(d_1) - \frac{50}{e^{.05\times 1}}N(d_2) + 50e^{-.05\times(1-.5)} \times N(-d_4) - 50 \times N(-d_3)$$

$$d_1 = \frac{\ln\left(\frac{50}{50}\right) + \left(.05 + \frac{1}{2}\times .16\right)\times 1}{.4\times\sqrt{1}} = .325$$

$$d_2 = .325 - .4\sqrt{1} = -.075$$

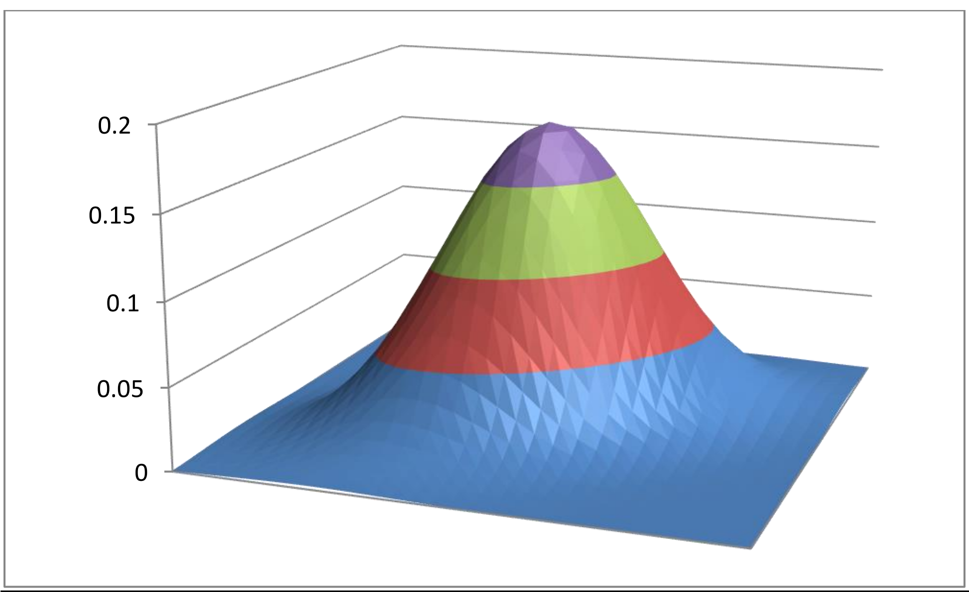
$$d_3 = \frac{\ln\left(\frac{50}{50}\right) + .05 \times 1 + \frac{1}{2} \times .16 \times .5}{.4 \times \sqrt{.5}} = .3182$$

$$d_4 = .3182 - .4 \times \sqrt{.5} = .0354$$

$$V_{chooser} = 50 \times .6274 - \frac{50}{e^{.05\times 1}} \times .4701 + 50e^{-.05\times .5} \times .4859 - 50 \times .3752 = 9.011 + 4.352$$
$$= 13.363$$

Appendix 12.A: Cumulative Standard Normal Bivariate Density Function

	-2	-1.8	-1.6	-1.4	-1.2	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
-2	0.01	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.01	0.01	0.01	0	0	0	0	0	0	0	0
-1.8	0.02	0.02	0.03	0.03	0.03	0.04	0.04	0.03	0.03	0.03	0.02	0.02	0.01	0.01	0.01	0	0	0	0	0	0
-1.6	0.02	0.03	0.03	0.04	0.05	0.05	0.05	0.05	0.05	0.04	0.03	0.03	0.02	0.01	0.01	0.01	0	0	0	0	0
-1.4	0.02	0.03	0.04	0.05	0.06	0.06	0.07	0.07	0.06	0.06	0.05	0.04	0.03	0.02	0.02	0.01	0.01	0	0	0	0
-1.2	0.02	0.03	0.05	0.06	0.07	0.08	0.09	0.09	0.09	0.08	0.07	0.06	0.05	0.03	0.02	0.02	0.01	0.01	0	0	0
-1	0.02	0.04	0.05	0.06	0.08	0.09	0.1	0.11	0.11	0.1	0.09	0.08	0.06	0.05	0.04	0.02	0.02	0.01	0.01	0	0
-0.8	0.02	0.04	0.05	0.07	0.09	0.1	0.12	0.13	0.13	0.13	0.12	0.1	0.09	0.07	0.05	0.04	0.02	0.02	0.01	0.01	0
-0.6	0.02	0.03	0.05	0.07	0.09	0.11	0.13	0.14	0.15	0.15	0.14	0.13	0.11	0.09	0.07	0.05	0.03	0.02	0.01	0.01	0
-0.4	0.02	0.03	0.05	0.06	0.09	0.11	0.13	0.15	0.17	0.17	0.17	0.15	0.13	0.11	0.09	0.06	0.05	0.03	0.02	0.01	0.01
-0.2	0.02	0.03	0.04	0.06	0.08	0.1	0.13	0.15	0.17	0.18	0.18	0.17	0.15	0.13	0.1	0.08	0.06	0.04	0.03	0.02	0.01
0	0.01	0.02	0.03	0.05	0.07	0.09	0.12	0.14	0.17	0.18	0.18	0.18	0.17	0.14	0.12	0.09	0.07	0.05	0.03	0.02	0.01
0.2	0.01	0.02	0.03	0.04	0.06	0.08	0.1	0.13	0.15	0.17	0.18	0.18	0.17	0.15	0.13	0.1	0.08	0.06	0.04	0.03	0.02
0.4	0.01	0.01	0.02	0.03	0.05	0.06	0.09	0.11	0.13	0.15	0.17	0.17	0.17	0.15	0.13	0.11	0.09	0.06	0.05	0.03	0.02
0.6	0	0.01	0.01	0.02	0.03	0.05	0.07	0.09	0.11	0.13	0.14	0.15	0.15	0.14	0.13	0.11	0.09	0.07	0.05	0.03	0.02
0.8	0	0.01	0.01	0.02	0.02	0.04	0.05	0.07	0.09	0.1	0.12	0.13	0.13	0.13	0.12	0.1	0.09	0.07	0.05	0.04	0.02
1	0	0	0.01	0.01	0.02	0.02	0.04	0.05	0.06	0.08	0.09	0.1	0.11	0.11	0.1	0.09	0.08	0.06	0.05	0.04	0.02
1.2	0	0	0	0.01	0.01	0.02	0.02	0.03	0.05	0.06	0.07	0.08	0.09	0.09	0.09	0.08	0.07	0.06	0.05	0.03	0.02
1.4	0	0	0	0	0.01	0.01	0.02	0.02	0.03	0.04	0.05	0.06	0.06	0.07	0.07	0.06	0.06	0.05	0.04	0.03	0.02
1.6	0	0	0	0	0	0.01	0.01	0.01	0.02	0.03	0.03	0.04	0.05	0.05	0.05	0.05	0.05	0.04	0.03	0.03	0.02
1.8	0	0	0	0	0	0	0.01	0.01	0.01	0.02	0.02	0.03	0.03	0.03	0.04	0.04	0.03	0.03	0.03	0.02	0.02
2	0	0	0	0	0	0	0	0	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.01



This table is truncated, depicting cumulative densities between -2.0 and 2.0, and with only significant digits to within .01. More complete tables and more accurate values can be found in the spreadsheet for the course, Knopf and Teall [2015], Drezner [1978] or in a variety of locations on the web.