Chapter 4: Decision Making Under Uncertainty

A. Expected Utility

In this chapter, we depart from the assumption of certainty and assume that decision-makers face uncertainty. Uncertainty is often conveniently modeled with the assistance of simple gamble descriptions, which might be illustrated as follows:

$$p \circ x_1 \oplus (1-p) \circ x_2 \in \mathcal{G}$$

This statement reads as follows: With probability $p \in [0,1]$, the decision-maker receives payoff (or consequence) x_1, and with probability $(1-p)$, receives payoff x_2, and that this gamble is an element in the space of gambles $\mathcal{G}$. We will assume the following where $q \in [0,1]$:

$$p \circ x_1 \ \oplus (1-p) \circ x_2 \sim (1-p) \circ x_2 \oplus p \circ x_1$$

$$q \circ (p \, o \, x_1 \ \oplus (1-p) \circ x_2) \oplus (1-q) \circ x_2 \ \sim qp \circ x_1 \ \oplus (1-\mathrm{q}p) \circ x_2$$

These two assumptions mean that the framing of the decision does not affect the desirability of the gamble and that consistent compounding of gambles does not affect their rankings.

<u>The St. Petersburg Paradox</u>

In 1713, the mathematician Nicholas Bernoulli reasoned that a rational gambler should be willing to buy a gamble for its expected value. For example, it seemed rational for a gambler to invest up to $1 for a gamble that paid either $2 or zero based on the toss of a coin. He extended his reasoning to a series of coin tosses, continuing to reason that the ultimate value of the more complex gamble should still be its expected value. His cousin, Daniel Bernoulli presented his paradigm in 1738 at a conference of mathematicians in St. Petersburg.[1] His extended problem, commonly referred to as the St. Petersburg Paradox, was concerned with why gamblers would pay only a finite sum for a gamble with an infinite expected value. Suppose, in Bernoulli's paradigm, the coin lands on its head on the first toss, the gamble payoff is $2. If the coin lands tails, it is tossed again. If the coin lands heads on this second toss, the payoff is $4, otherwise, it is tossed a third time. If the coin lands heads on the third toss, the payoff doubles again to $8; otherwise, it is tossed again for a potential payoff of $16. The process continues until the payoff is determined by the coin finally landing heads. Where *n* equals infinity, the expected value of this gamble is determined by the following equation:

$$E[V] = (.5^1 \times 2^1) + (.5^2 \times 2^2) + (.5^3 \times 2^3) + \ldots + (.5^n \times 2^n)$$

This equation is based on the expectation that the probability of the coin landing heads on the first (or any) toss equals .5. If the coin lands heads on the first toss, the payoff equals $\$2 = 2^1$. Since there is a fifty percent chance the coin will land tails on the first toss and a fifty percent chance the coin will land heads on the second toss, the probability of achieving a payoff of $4 = 22 on the second toss is $.5 \cdot .5 = .25$. Thus, the probability of having a payoff equal to $2^n = .5^n$.

[1] Nicholas Bernoulli first proposed this problem in a letter to Pierre Raymond de Montmort dated 9 September 1713, who then published it in his book later that year. The Swiss mathematician Gabriel Cramer actually proposed an essentially identical solution ten years before Daniel. Correspondence between Nicholas Bernoulli and Cramer is available at http://www.cs.xu.edu/math/Sources/Montmort/stpetersburg.pdf#search=%22Nicolas%20Bernoulli%22. Daniel further argued in his essay that risk-averse investors should diversify.

The expected value of the gamble equals the sum of all potential payoffs times their associated probabilities. So, exactly what is the expected value of this gamble? We simplify the equation above as follows:

$$E[V] = (.5^1 \times 2^1) + (.5^2 \times 2^2) + (.5^3 \times 2^3) + \ldots + (.5^n \times 2^n)$$
$$E[V] = (.5 \times 2) + (.5 \times 2) + (.5 \times 2) + \ldots + (.5 \times 2)$$
$$E[V] = (1) + (1) + (1) + \ldots + (1)$$

It appears, since there is some possibility that the coin is tossed tails an infinity of times, the expected or actuarial value of this gamble is infinite. This seems quite obvious from a mathematics perspective. Paradoxically, Bernoulli found that none of the esteemed mathematicians at the conference would be willing to pay an infinite sum (or, in most cases, even a large sum) of money for the gamble with infinite actuarial value. Were the mathematicians simply irrational? Or, should the worth or market value of a gamble or investment be less than its actuarial or expected value.

Bernoulli opined that the resolution to this paradox is the now commonly accepted notion of “diminishing marginal utility,” which holds that as the wealth of a person increases, the satisfaction that he derives increases, but at a lesser rate (See Figure 1). More money produces more satisfaction, but the rates of increase in satisfaction are less than the rates of increase in wealth. So the worth of a gamble to an investor is less than its expected value because the utility derived from each dollar of potential gains is less than the utility of each dollar potentially lost. Bernoulli proposed a log-utility function where an individual’s level of satisfaction derived from wealth is related to the log of his wealth level. The key to this utility function is that satisfaction increases as wealth increases, but at a lesser rate. This means that, an investor stands to lose more satisfaction in an actuarially fair gamble than he stands to gain. The potential loss in a “double-or-nothing” bet is more significant than the potential gain. Thus, investors will reject actuarially fair gambles because, on average, they lose satisfaction of utility.

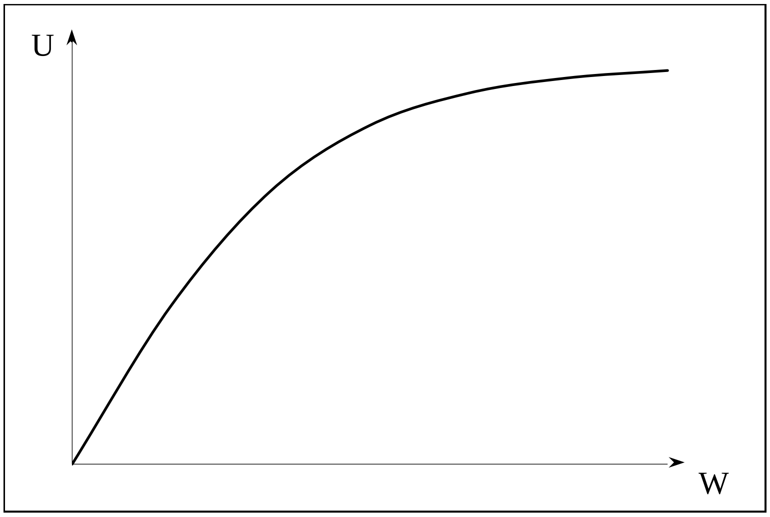

Figure 1: Utility of Wealth

The implication of the utility function is that rational investors should seek to maximize the expected utility of their wealth, not their expected wealth itself. Furthermore, this theory of utility can serve as the theoretical foundation for risk aversion. Thus, rational investors can be motivated not only by greed, but by fear as well.

Axioms of Choice: von Neumann and Morgenstern

Recall our discussion from Chapter 1 concerning the set of axioms offered by John von Neumann and Oscar Morgenstern [1947]. We will use the same of axioms to derive the Expected Utility Hypothesis. We start by assuming that decision-makers seek to identify and select payoffs x_i from a convex subset X so as to achieve maximal satisfaction. The element x_i represents a payoff can be selected by the decision-maker from the *n* elements of X. The first three axioms ensure decision-maker rationality:

1. *Reflexivity*: For an entire set X of payoffs x_i, $x_j \gtrsim x_j$ (x_j is at least as desirable as x_j or x_j is weakly preferred to x_j). This axiom might be regarded as merely a formal mathematical necessity.
2. *Completeness (or Comparability)*: For an entire set of payoffs x_i, either $x_j \succ x_k$ (x_j is strictly preferred to x_k), $x_j \prec x_k$ ((x_j is less desirable than x_k) or $x_j \sim x_k$ (x_j is equally desirable to x_k) for all *j* and *k*. Thus, the decision-maker can fully identify and specify his preferences over the entire set of commodities.
3. *Transitivity*: For any x_i, x_j, x_k, if $x_i \succ x_j$ and $x_j \succ x_k$, then $x_i \succ x_k$. This axiom ensures consistency among choices.

While the three axioms listed above are sufficient to ensure decision-maker rationality, working with such preference relations can be difficult at best when *n* is very large. Hence, it is useful to develop and apply a rule that assigns values to choices. Such a rule might be a cardinal utility function. A cardinal utility function assigns a unique number (utility level) to each and every choice among payoffs. The utility function is simply a convenient tool for comparing preferences. Three more axioms are needed to establish a cardinal utility function and the Expected Utility Paradigm:

1. *Strong Independence*: If $x_i \sim x_k$, then for any $p \in [0, 1]$, $px_i + (1\text{-}p)x_j \sim px_k + (1\text{-}p)x_j$. It may be useful to interpret p as a probability for uncertain outcomes. This axiom implies that preference rankings are not affected by inclusion in more complicated arrangements. Independence implies that the choices that the participant in the gamble makes in one state of nature are the same as he would make in some other state of nature. That is, the outcome of the gamble will not affect other choices made by the gambler.
2. *Measurability (Continuity or Intermediate Value)*: If $x_i \succ x_j \succ x_k$, then there exists some *p* such that $px_i + (1\text{-}p)x_k \sim x_j$. This implies non-existence of lexicographic (dictionary) orderings. Lexicographic orderings (such as safety first criteria) imply discontinuities in utility functions.
3. *Ranking*: Assume that $x_i \succ x_j \succ x_k$ and $x_i \succ x_m \succ x_k$, and $x_j \sim px_i + (1\text{-}p)x_k$ and $x_m \sim \gamma x_i + (1\text{-}\gamma)x_k$ where $p, \gamma \in [0,1]$. Then it follows that if $p > \gamma$, $x_j \succ x_m$ or if $p = \gamma$, $x_j \sim x_m$.

These six axioms are sufficient to construct a cardinal utility function where utility can be represented with numbers. We will usually add two more assumptions to this list:

1. *Greed*: (*local non-satiation*): If $x_i \geq x_j$, $x_i \succ x_j$. Decision-makers prefer more wealth to less.

2. *Diminishing marginal utility* (*convexity*): This assumption need not always apply, but it does seem realistic.

<u>The Expected Utility Paradigm</u>

We will define a utility function U: $\mathcal{G} \to$ /representing $\succsim$ on $\mathcal{G}$ such that $U(g)$, $g \in \mathcal{G}$ is a utility number assigned to gamble g. Now we define the *Expected Utility Property*:

The utility function U(g): $\mathcal{G} \to$ /has the expected utility function if there is an assignment of numbers ($U(x_1)$, ..., $U(x_n)$) to the n outcomes such that for every simple gamble we have:

$$U(g) = \sum_{i=1}^{n} p_i\, U(x_i) = E[U(g)]$$

where $(p_1\ o\ x_1 \oplus \ldots \oplus p_n\ o\ x_n)$ *is the simple gamble* g *and* $(p_1 + \ldots + p_n = 1)$.

A utility function with this expected utility form is called a von Neumann-Morgenstern (VNM) expected utility function. An expected utility maximizer seeks to maximize E[U(g)].

The *Existence Theorem* for the VNM Utility Function on $\mathcal{G}$ is as follows:

Where preferences $\succsim$ on $\mathcal{G}$ satisfy Axioms 1 through 6 above, there exists a utility function U: $\mathcal{G} \to$ /representing $\succsim$ on $\mathcal{G}$ such that U(g) has the expected utility property.

This means that as long as preferences satisfy the axioms listed above, there will exist an expected utility function that represents the gambler's preferences.[2] This expected utility function will be unique to a *positive affine transformation*. This means that if utility function U were transformed as follows: $V = aU + b$, the properties of U, including rankings, would apply to utility function V.

B. Risk Aversion, Large Risks and Insurance

For the typical investor utility of wealth function, we might expect to have $U(W)$ where $U'(W) > 0$ and $U''(W) < 0$. This will be consistent with $U(E[W]) > E[U(W)]$. The Certainty Equivalent (CEQ_{W+z}) associated with a given actuarially neutral gamble z at wealth level W is given as follows:

$$CEQ_{w+z} = U^{-1}(E[U_{W+z}])$$

where U^{-1} is the inverse function of the utility function and E[U_{W+z}] is the expected utility associated with wealth level w with gamble z. Markowitz defined a utility maintenance risk premium for an actuarially neutral gamble with expected wealth level E[W+z] as follows:

$$\pi_z = E[W+z] - CEQ_{W+z}$$

Figure 2 depicts these relationships involving the Markowitz risk premium.

[2] This existence theorem is proven in Varian [1992].

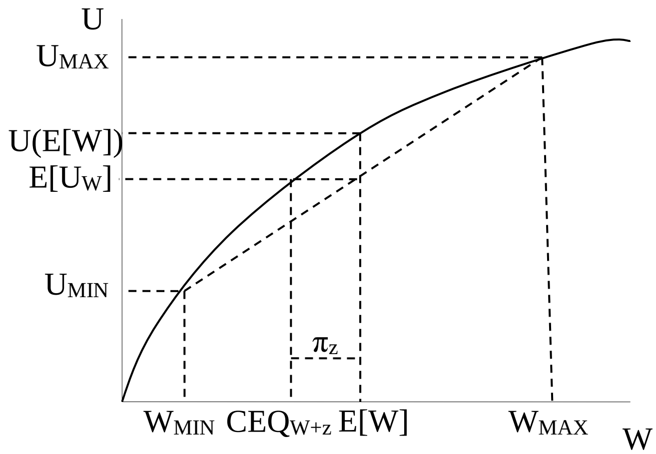

Figure 2: CEQ and Expected Utility of Wealth

As discussed earlier, an individual who prefers more wealth to less will have an upward sloping utility of wealth function. His function will be concave down if he has diminishing marginal utility with respect to wealth. We will demonstrate here that this type of individual is also risk averse, though we will focus on larger risks. Most investors will probably prefer more to less and will be risk averse.

Illustration: Square Root Utility Function

In the previous part of the Utility of Wealth Application example, we defined a utility of wealth function for a particular individual as follows:

$$U = .5 \cdot \sqrt{W}$$

We find the first derivative of the utility function as follows:

$$f'(W) = .25\, w^{-1/2} = .25\frac{1}{\sqrt{W}} > 0$$

This derivative is positive, indicating that utility increases. The second derivative of the utility function is found:

$$f''(w) = -.125W^{-3/2} = -.125\frac{1}{\sqrt{W^3}} < 0$$

Thus, this utility function is concave down indicating diminishing marginal utility with respect to wealth.

Consider an actuarially fair gamble where an individual with Utility Function has a fifty percent probability of receiving $20,000 and a fifty percent chance of receiving nothing. Suppose that the individual has no other wealth. The expected value of this gamble is $10,000. If the individual wins $20,000, Outcome one is realized and his utility level will be 70.7:

$$U = .5 \times \sqrt{20{,}000} = 70.7 \quad .$$

If the individual wins nothing, Outcome two is realized and his utility level will be zero:

$$U = .5\sqrt{0} = 0 \quad .$$

Therefore, the expected utility of the gamble is 35.35:

$$E(U) = \sum_{i=1}^{n} p_i u_i$$

$$E(U) = .5 \times (.5 \times \sqrt{20{,}000}) + .5 \times (.5 \times \sqrt{0}) = 35.35$$

However, we have already found that the expected utility of $10,000 is 50. Because the gamble represents a reduction in utility from the certain sum of $10,000, this individual will not pay $10,000 for the gamble. In fact, this gamble is worth only $5,000 to the individual, determined by solving the above equations for *W* to obtain CEQ:

$$\text{CEQ} = (\text{E}[\text{U}]/.5)^2$$

since $\text{E}[U] = .5 \times \sqrt{W}$. Thus,

$$CEQ = (35.35/.5)^2 = 70.7^2 = \$5000 \quad .$$

This individual with diminishing marginal returns with respect to wealth is risk averse. Therefore, he will not pay as much for a gamble as he would for a certain sum with the same expected value. If this individual did pay $10,000 for this gamble, he would find that his potential increase in utility associated with winning $20,000 would be more than offset by the potential decrease in utility associated with winning nothing while losing his initial investment. This individual dislikes risk to the extent that he will pay only $5,000, his certainty equivalence, for a gamble with an expected value of ten thousand dollars. This is the type of individual who would purchase insurance against potential losses.

Consider a second individual with the same utility of wealth function owning a lottery ticket with a 50% chance of paying nothing and a 50% chance of paying twenty thousand dollars. This individual also has twenty thousand dollars in cash. His expected utility level is 85.35:

$$\begin{aligned}\text{E}(U) &= .5 \times (.5 \times \sqrt{0 + 20{,}000}) + .5 \times (.5 \times \sqrt{20{,}000 + 20{,}000}) \\ &= [.25 \cdot 141.4] + [.25 \cdot 200] = 85.35\end{aligned}$$

The relevant values are depicted in Figure 3. This second individual has a fifty percent chance of attaining a terminal wealth level of $20,000 and a fifty percent chance of attaining a terminal wealth level of $40,000. Thus, his expected terminal wealth and utility levels are $30,000 and 85.35. Of course, the actual wealth and utility levels will differ from their expected levels.

Nonetheless, there exists some certain level of wealth with exactly the same utility level as the gamble. This certain level can be solved by solving the utility function for his certainty equivalence:

$$CEQ = (85.35 / .5)^2 = \$29,136$$

Therefore, the second individual would be as satisfied with $29,136 as with his current uncertain wealth holdings. Since the expected value of his terminal wealth level is $30,000, he would pay up to $864 to insure his gamble at $10,000. That is, the second individual would pay $864 to ensure that he receives a certain $10,000 from the lottery ticket (its expected value) rather than face the prospect of receiving either nothing or receiving $20,000. Thus, the maximum premium one will pay for insurance guaranteeing some wealth level gamble can be found as follows:

$$\pi = E[W] - CEQ$$

$$\pi = \$30,000 - \$29,136 = \$864$$

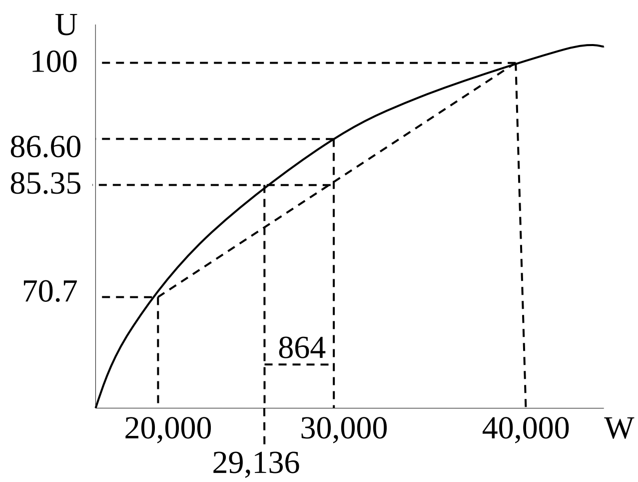

Figure 3: CEQ Illustration and Expected Utility of Wealth

C. Risk Aversion, Small Risks and Insurance

One of the most important concepts for analyzing individual preferences with respect to risk is the concave utility function. *Jensen's Inequality*, a concept widely used in many areas of math, concerns some concave function of a random variable. Jensen's Inequality applies to investors who have diminishing marginal utility with respect to wealth:

If x is a random variable and f(x) is a strictly concave function of x, Ef(x) < fE(x).

To demonstrate this inequality, we will only concern ourselves with continuous differentiable utility functions as are consistent with the axioms presented earlier. Strict concavity of the function f implies that $f''(x) < 0$. Suppose that $f'(x) > 0$ and $E[x] = k$. Then, by definition of strict concavity, $f(x) < f(k) + f'(k)(x - k)$. Applying the expectations operator to both sides, we have $Ef(x) < f(k) + f'(k)(E[x] - k) = f(k) = fE(x)$.

Jensen's Inequality implies that, given a particular actuarially fair uncertainty, the expected utility of wealth is less than the utility of expected wealth:

$$\int U(W)dF(W) \le U\left(\int WdF(W)\right) \; \forall \; F(W),$$

where $dF(W)$ is the density associated with a given level of wealth. This means that the actuarially fair gamble reduces the gambler expected utility without reducing his expected wealth. Figure 4 depicts a risk averse investor's utility of wealth function, where it is apparent that $U[E[W]] < E[U_W]$. When gamblers (investors) have diminishing marginal utility with respect to wealth, they will be risk averse. This means that they will prefer certainty to uncertainty and will give up wealth in order to reduce risk. This section is concerned with how much wealth investors will be willing to sacrifice for increased certainty and what will be certainty equivalents of uncertain wealth levels. Because marginal utilities with respect to wealth are unlikely to be constant, we will need to distinguish between small and large risks.

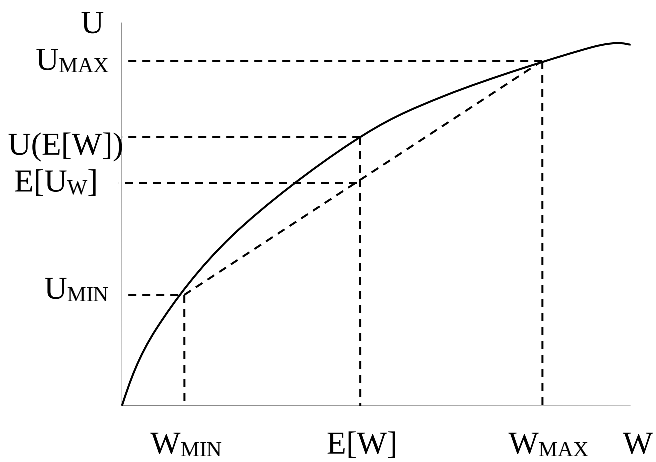

Figure 4: Expected Utility of Wealth and Utility of Expected Wealth

Risk Aversion in the Small

In the previous section, we discussed utility as a function of wealth. This application is concerned with utility in a setting of uncertainty along with the measurement of investor risk aversion. Since one might expect that an investor is likely to prefer certainty to uncertainty, one might expect that he would require a premium to accept a risk of a given level (or pay a premium to eliminate a given risk). The higher the premium that an investor would require to accept a given risk, the more risk averse we can infer that he is.

Assume that the investor selects his investment so as to maximize the expected utility level that he associates with his level of wealth W.[3] Also, assume that the investor's wealth level is subject to some small level of uncertainty represented by z whose expected value is zero. Thus, z represents an actuarially fair gamble or random number that can assume any value, but has an expected value equal to zero. Assume that the investor is averse to risk and would be willing to pay a premium π to eliminate this risk. Our problem here is to determine the maximum premium that he would be willing to pay; we will use the level of this premium to measure the investor's level of risk aversion. First, we note that the maximum premium that the investor is willing to pay would be that which equates the utility associated with his current uncertain level of wealth

[3] See Pratt [1964]

with the level of utility he would realize if he "bought insurance" and eliminated his risk:

$$E[U(W + z)] = U(W - \pi)$$

Thus, expected utility is currently a function of the current level of wealth and the gamble; if the gamble is eliminated, utility will be a function of the current wealth level minus the insurance premium. Our problem is to solve this equality for π. If our utility function is at least twice differentiable, we solve by performing a Taylor series expansion around both sides of the equality:

$$E[U(W) + zU'(W) + \tfrac{1}{2} z^2 U''(W) + \ldots] = U(W) - \pi U'(W) + \ldots..$$

Since E[z] = 0, σ_z^2 = E[z^2] and E[z]U'(W) can be dropped from the equality. Following convention, we will approximate by dropping all of the left-hand side higher order terms not explicitly stated in the above equality. This convention is quite reasonable if we are willing to assume that the risk is normally distributed, meaning that E[z^3] will equal 0. We will also approximate the right-hand side of the utility function above by dropping all terms and derivatives of higher order than one to obtain:

$$E[U(W)] + \frac{1}{2}\sigma_z^2 \; U''(W) = U(W) - \pi U'(W)$$

Now, we solve for the risk premium as follows:

$$\pi = -\frac{1}{2}\sigma_z^2\left[\frac{U''(W)}{U'(W)}\right]$$

When used in this context, *-U''(W)/U'(W)* is referred to as the *Arrow-Pratt Absolute Risk Aversion Coefficient* (*ARA*), which indicates an investor's aversion to a given risk σ_z^2, based on his utility of wealth function *U(w)* and his current level of wealth. This absolute risk aversion coefficient measure generalizes beyond normally distributed risks.

A given investor *A* will accept a particular gamble that is unacceptable to another investor *B* if his *ARA* is smaller at their current wealth levels:

$$-\frac{U_A''(Ww)}{U_A'(W)} < -\frac{U_B''(W)}{U_B'(W)}$$

In this scenario, Investor *B* is more risk averse than is Investor *A*; his utility of wealth function exhibits more concavity than does the utility curve for Investor *A*. Pratt's theorem holds that each of the following statements is equivalent to the inequality above:

$G(U_A(W)) = U_B(W)$ for some strictly concave function G
$\pi_A(z) < \pi_B(z)$ for all z with E[z] = 0

Thus, Investor *B*'s utility of wealth function is some strictly concave function of that for Investor

A and Investor *B* will be willing to pay more to insure away some actuarially fair gamble than Investor *A*. All three of the inequalities equivalently imply that Investor *B* is more risk averse than Investor *A*. This is an informal presentation of *Pratt's Theorem*.

Absolute risk aversion (*ARA*) concerns how investors react to gambles of given monetary size. *Relative risk aversion* (*RRA*) concerns how investors react to gambles relative to their overall wealth levels:

$$RRA = ARA \times W = -\frac{U''(W)W}{U'(W)}$$

Generally, one might expect that as wealth increases, *ARA* will decrease and *RRA* will either decrease or remain constant. Decreasing *ARA* with respect to wealth suggests that investors are willing to take larger monetary risks as their wealth levels increase. Decreasing *RRA* with respect to wealth suggests that investors are willing to take larger proportional risks as their wealth levels increase. However, each utility function should be tested to determine whether it is appropriate for the given investor's preferences.

Illustration: Quadratic Utility and Risk Aversion

At least over a relevant range, quadratic utility functions can be structured to be consistent with both greed (more is preferred to less) and diminishing marginal utility:

$$U(w) = aW - bW^2 \quad \text{for } a, b > 0;\ a > 2bW$$

Note first that the first and second derivatives of utility with respect to wealth are $U'(W) = a - 2bW$ and $U''(W) = -2b$. Quadratic utility has a number of desirable properties, most importantly that it can be rewritten to express utility as a function of expected wealth E[W] and the variance of wealth σ_W^2. Define the variance of wealth as follows:

$$\sigma_W^2 = E[W - E[W]]^{\,2}$$

Next, rewrite the variance as follows:

$$\sigma_W^2 = E[W^2] - 2E[W]E[W] + E[W]^{\,2} = E[W^2] - [E[W]]^{\,2}$$

which implies that:

$$E[W^2] = \sigma_W^2 + [E[W]]^{\,2}$$

The Expected utility of wealth function can be written as follows:

$$E[U(w)] = aE[W] - bE[W^2]$$

$$E[U(W)] = aE[W] - b\sigma_W^2 - b[E[W]]^{\,2}$$

Thus, expected utility is easily expressed as a function of expected wealth and the variance of wealth when utility is a quadratic function of wealth. Derivatives of expected utility with respect to expected wealth and the variance of wealth are as follows:

$$\frac{\partial \mathrm{E[U(W)]}}{\partial \mathrm{E[W]}} = a - 2bW > 0 \quad for\ a > 0, a > 2bW$$

$$\frac{\partial \mathrm{E[U(W)]}}{\partial \sigma_W^2} < 0 \quad for\ b > 0, b < a/2W$$

Clearly, utility increases in expected wealth and is decreasing in the variance associated with wealth. Absolute and relative risk aversion coefficients are computed as follows:

$$ARA = -\frac{U''(W)}{U'(W)} = \frac{2b}{a - 2bW}$$

$$RRA = ARA \times W = -\frac{U''(W)W}{U'(W)} = \frac{2bW}{a - 2bW}$$

Sensitivities of ARA and RRA to wealth levels are computed with the Quotient Rule as follows:

$$\frac{\partial ARA}{\partial \mathrm{W}} = \frac{4b^2}{(a - 2bW)^2} > 0$$

$$\frac{\partial RRA}{\partial \mathrm{W}} = \frac{4b^2}{(a - 2bW)^2} + \frac{2b(a - 2bW)}{(a - 2bW)^2} = \frac{2b}{(a - 2bW)^2} > 0$$

These increasing marginal absolute and absolute risk aversion coefficients suggest that as investors' wealth increases, their propensities to take on increased absolute and proportional risks decrease. This characteristic seems somewhat unrealistic for most investors. Thus, quadratic utility functions are often undesirable in analyses involving changes in wealth levels.

D. Risk Aversion and Portfolio Allocation

Investor portfolio allocation might be expected to be a function of investor risk aversion (obtained through the utility function), and the risk and return levels of available assets. Here, we explore how portfolio allocation between risky and risky assets might be determined as a function of the investor's utility in a single time-period framework.

Illustration: Constant Absolute Risk Aversion (CARA)

Constant absolute risk aversion (CARA) functions imply that risk aversion is constant with respect to wealth such that investors are willing to take on the same level of risk irrespective of their wealth levels. Suppose that an investor's utility of terminal consumption c_T is the computed in a one-period framework with the following CARA function:

$$U(c_T) = -\alpha^{-1} e^{-\alpha c_T}$$

where α is a constant. Note that the investor's marginal utility of consumption, its rate of change given changes in consumption and the absolute risk aversion coefficients are given by the following:

$$U'(c_T) = e^{-\alpha c_T}$$

$$U"(c_T) = -\alpha e^{-\alpha c_T}$$

$$ARA = \frac{U"(c_T)}{U'(c_T)} = -\alpha$$

Notice that the investor's absolute risk aversion coefficient -α is a simple constant. Suppose that the investor seeks to invest some level S_0 of his wealth W_0 in risky assets. Thus, assume an initial wealth level W_0 such that terminal consumption is calculated from security returns as follows:

$$c_T = S_0\tilde{r} + (W_0 - S_0)r_f = W_0 r_f + S_0(\tilde{r} - r_f)$$

where the risky security has a normally distributed random return $\tilde{r}$ and the remainder is invested in the risk free security with return r_f. Expected utility of terminal consumption is computed as follows:

$$E[U(c_T)] = E\left[-\alpha^{-1}e^{-\alpha\left(W_0 r_f + S_0(\tilde{r}-r_f)\right)}\right] = -\alpha^{-1}e^{-\alpha W_0 r_f}E\left[e^{-\alpha S_0(\tilde{r}-r_f)}\right]$$
$$= -\alpha^{-1}e^{-\alpha W_0 r_f}e^{-\alpha(S_0 E(\tilde{r}-r_f) - \alpha S_0^2 \frac{\sigma^2}{2})}$$

The right side of this expected utility function follows from the assumption of a normal distribution for the random return $\tilde{r}$ and risk premium $(\tilde{r} - r_f) \sim N\left[E(\tilde{r} - r_f), \sigma^2\right]$ such that:[4]

$$E\left[e^{(\tilde{r}-r_f)}\right] = e^{\left(E(\tilde{r}-r_f)+\frac{\sigma^2}{2}\right)}$$

The individual seeks to maximize his utility of terminal consumption with respect to the control variable S_0, the individual's optimal portfolio choice. The investor will decide how much wealth S_0 to invest in the risky asset. This problem is the same as working through the following:

$$\frac{\partial}{\partial S_0}\left[S_0 E(\tilde{r} - r_f) - \alpha S_0^2\,\frac{\sigma^2}{2}\right] = 0$$

which has the results:

$$E(\tilde{r} - r_f) - \alpha\frac{2S_0\sigma^2}{2} = 0$$

[4] The risk premium is a Brownian motion $(\tilde{r} - r_f) \sim N\left[E(\tilde{r} - r_f), \sigma^2\right]$. This implies that:

$$E\left[e^{(\tilde{r}-r_f)}\right] = e^{\left(E(\tilde{r}-r_f)+\frac{\sigma^2}{2}\right)}$$

$$S_0 = \frac{E(\tilde{r} - r_f)}{\alpha\sigma^2}$$

We see that the investor invests more money in the risky asset as returns on the risky asset increase and as its risk declines. However, it is important to note that the investor's monetary investment in the risky asset is unaffected by initial wealth. Hence, we refer to the particular utility function used in this illustration to be a constant absolute risk aversion (CARA) function, which, generally, is a somewhat unrealistic utility function because it implies that all investors subject the same absolute amount to risky investment.

E. Insurance and Co-Insurance

Now, we will consider insurance, pricing of insurance and the insured's option to provide coinsurance. Suppose that a consumer with a utility function U has a wealth level equal to W if no loss occurs. However, a loss of L will occur with probability p. The consumer can purchase insurance on fraction α of this loss L with a premium of $\pi = pcL$. The coefficient c can be thought of as the premium markup; if $c = 1$, the insurance is priced to be actuarially fair. If $c > 1$, the insurance company expects to obtain a profit on the sale of the policy. Thus, each unit of insurance will cost pc, while the dollar value of the loss, if it occurs, is L. Since the consumer can choose the proportion α of the potential loss L that he might incur, his coinsurance is $(1-\alpha)L$. If the consumer incurs a loss equal to L, the insurance policy will pay her αL. How much insurance α should the consumer purchase given the probability p of a loss L?

The consumer should select a level of insurance α so as to maximize her expected utility:

$$\underset{\alpha \geq 0}{MAX} E[U] = \underbrace{(1\text{-}p) \cdot U[W - pcL\alpha]}_{\substack{\text{Utility of Wealth} \\ \text{in the Absence of a Loss}}} + \underbrace{p \cdot U[W - L - pcL\alpha + L\alpha]}_{\substack{\text{Utility of Wealth if a Loss} \\ \text{is Incurred}}}$$

This expression might be interpreted as follows: The insured selects the level of co-insurance α such that her expected utility is maximized. To solve for the optimal level of insurance α^*, we will use the Chain Rule to solve for the derivative of $E[U]$ with respect to α as follows:

$$\frac{\partial E[U]}{\partial \alpha} = -pcL(1\text{-}p)U'[W - pcL\alpha^*] + (L\text{-}pcL)pU'[W - L - pcL\alpha^* + L\alpha^*] \leq 0$$

The inequality exists because the cost of the insurance might be actuarially unfair ($c > 1$) and prohibitively expensive to purchase in any quantity. Since the consumer cannot purchase negative quantities of insurance ($\alpha \geq 0$), this maximization problem is constrained. The derivative will equal zero if the cost pc per unit of insurance α is sufficiently low.

Suppose, for example, that the insurance pricing is actuarially fair; that is, $\pi = \alpha pcL$ and $c = 1$. What is the optimal level of coinsurance? We will solve for α^* in the following from the derivative above and simplify:

$$pcL(1\text{-}p)U'[W - pcL\alpha^*] = (L\text{-}pcL)pU'[W - L - pcL\alpha^* + L\alpha^*]$$

$$U'[W - pL\alpha^*] = U'[W - L - pL\alpha^* + L\alpha^*]$$

$$[W - pL\alpha^*] = [W - L - pL\alpha^* + L\alpha^*]$$

$$0 = [-L + L\alpha^*]$$

$$\alpha^* = 1$$

This result implies that the risk-averse consumer will fully insure if insurance is priced to be actuarially fair. However, we can expect that co-insurance will increase as the premium increases.

Illustration: Insurance and Coinsurance

Suppose that a consumer with a quadratic utility function $U = aW - bW^2 = 20W - .005W^2$ has a wealth level equal to 1,000 if no loss occurs. However, further suppose that the consumer's wealth is subject to a potential loss of 500 from her wealth, with probability $p = .25$. The consumer can purchase insurance on any fractional amount $0 \leq \alpha \leq 1$of this gamble with a premium based on $\pi = pcL$. For example, if the investor insures fraction $\alpha = .8$ or 400 of this gamble, the most that she can lose is 100, her coinsurance amount. Finally, suppose that the mark-up on insurance is $c = 1.02$. Thus, each unit of insurance costs $pc = .255$, and up to 500 units can be purchased. Thus, insurance on 400 of the loss costs $\pi = \alpha pcL = .8\times.25\times1.02\times500 = 102$.

Clearly, the insurance policy is not actuarially fair because $c > 1$ and because the premium on the full amount of the loss, $\pi = pcL = 127.5$, exceeds the expected value of the loss $E[L] = pL = 125$. The consumer will determine proportion α by substituting for α in the first order condition from above using the quadratic utility function defined in the previous paragraph:

$$-pcL(1-p)U'[W - pcL\alpha^*] + (L-pcL)pU'[W - L - pcL\alpha^* + L\alpha^*] = 0$$

Since, in this quadratic utility illustration, $U'[W - pcL\alpha] = a - 2b[W - pcL\alpha]$ and $U'[W - L - pcL\alpha + L\alpha] = a - 2b[W - L - cpL\alpha + L\alpha]$, we can rewrite this first order condition as:[5]

$$(1-p)\cdot[-cpLa - 2b\alpha(cpL)^2 + 2bcpLW] + p\cdot[a(L-cpL) - 2b(L-cpL)^2\alpha - 2b(L-cpL)(W-L)] = 0$$

Next, we substitute in coefficients from our specific quadratic utility function and simplify:

$$.75\cdot[-1.02\cdot.25\cdot500\cdot20 - 2\cdot.005\cdot\alpha\cdot127.5^2 + 2\cdot.005\cdot1.02\cdot.25\cdot500\cdot1000]$$
$$+.25[20(500-1.02\cdot.25\cdot500) - 2\cdot.005\cdot(500-1.02\cdot.25\cdot500)^2\cdot\alpha - 2\cdot.005(500-1.02\cdot.25\cdot500)(1000-500) = 0$$

$$.75\cdot[-2550 - .01\alpha(127.5)^2 + 1275] + .25\cdot[20(500-127.5) - .01(372.5)^2\alpha - .01(372.5)(500)] = 0$$
$$.75\cdot[-2550 + 1275] + .25\cdot[20(500-127.5) - .01(372.5)(500)] = [.01\times(127.5)^2 + .01\times(372.5)^2]\alpha$$

$$-956.25 + 1396.875 = (121.921875 + 468.8125)\alpha *$$

Simplifying further, we finish solving for α* as follows:

[5] Based on this particular quadratic utility of wealth function, $U' = a - bW = 20 - .01W$.

$$\alpha * = \frac{-956.25 + 1396.875}{121.921875 + 468.8125} = .939874683$$

Thus, the consumer should insure proportion α^* = .939874683, calculated by substituting for α (or in this illustration, algebraically solving for α). The total premium for this insurance will be $\alpha\pi = \alpha pcL$ = 119.83, representing a profit of $\alpha\pi - \alpha pL$ = 119.83 - 117.48 = 2.35 for the insurance company. Notice that co-insurance (1-α) will increase as the cost per unit of insurance *c* increases. For example, repeating the calculations above for *c* = 1.03 would decrease α^* to .9097269 and the insured will raise her co-insurance level from (1-α) = .0601 to .0903. Co-insurance will tend to increase for risk-averse consumers as the cost per unit of insurance increases, as risk aversion decreases, as potential losses decrease and as the probability of a loss decreases.

F. Stochastic Dominance

Many types of portfolio and investment selection models make assumptions regarding either the form of probability distribution of returns faced by investors or about the form of investor utility of wealth functions. For example, the Capital Asset Pricing Model assumes either that security returns are normally distributed or that investors have quadratic utility functions. In reality, measurement of investor utility functions is, at best, extremely difficult. Determining the actual probability distribution of security returns is usually either difficult or impossible. Thus, portfolio selection may be aided by a set of rules that does not rely on determination of the exact return distribution and requires only the most essential information regarding investor preferences. The concept of stochastic dominance is such an example. It does not rely excessively on the exact form of investor utility functions and it does not necessarily require that return distribution functions be fully specified. Thus, stochastic dominance may be a useful portfolio and investment selection tool when we are able to make only the barest of assumptions or observations regarding utility and probability functions.

In portfolio analysis, a portfolio is considered dominant if it is not dominated by any other portfolio. One portfolio is considered to dominate a second portfolio if, from a given perspective or based on specific criteria, its performance is at least as good as the second portfolio under all circumstances (or states of nature) and superior under at least one circumstance. For example, first order stochastic dominance exists where one security has at least as high a payoff under each potential state of nature and a higher under at least one state. Table 1 lists three orders of stochastic dominance and the circumstances under which each might be used as a portfolio selection rule. In Table 1, U(w) designates the utility of wealth function, and A(w) represents the absolute risk aversion coefficient defined as follows:

$$ARA(w) = -\frac{U''(w)}{U'(w)}$$

Order of Stochastic Dominance	Used by Investors When
First order	More is preferred to less: U'(w) > 0
Second order	Safety is preferred to risk: U"(w) < 0
Third order	Investors have decreasing absolute risk aversion: A'(w) = {[U"(w)] ÷ [U'(w)]}^{2} - {U'"(w) ÷ U'(w)} < 0

Table 1
Orders of Stochastic Dominance

It is quite reasonable to assume that investors prefer more to less. Thus, whenever one asset exhibits first order stochastic dominance (defined below) over a second asset, the first asset will be preferred. Whenever an investor is risk averse and prefers more to less, an asset that exhibits second order stochastic dominance over a second will be preferred. Similarly, whenever an investor has decreasing risk aversion with respect to wealth, he is risk averse and he prefers more to less, an asset which exhibits third order stochastic dominance over an alternative asset will be preferred.

Suppose that there exist two assets *f* and *g* whose payoffs *f*(x) and *g*(x) are dependent on some ordered random variable x such that $f'(x)>0$ and $g'(x)>0$. Thus, as the value of random variable x increases, the payoffs on securities *f* and *g* increase. We will not specify the exact characteristics of individual investor utility functions; we state only that investors will prefer a higher payoff to a lower payoff. The probability distribution functions $P_f(x)$ or $P_g(x)$ can be used to represent the probability that security payoffs x will be less than or equal to some constant x^*. Define the following probability distribution functions for payoffs on securities *f* and *g*:

$$P_f\ (x^*) = \int_{-\infty}^{x^*} p_f\,(x)dx$$

$$P_g\ (x^*) = \int_{-\infty}^{x^*} p_g\,(x)dx$$

Security *g* is said to exhibit first, second or third order stochastic dominance over security *f* if the appropriate conditions from Table 2 hold.

Table 2

Order of Stochastic Dominance	Conditions
First order	$P_f(x) \geq P_g(x)$ for all *x* U'(w) > 0 $P_f(x) > P_g(x)$ for some *x*
Second order	$\int_{-\infty} P_f(x)dx \geq \int_{-\infty} P_g(x)dx$ for some *x* $\int_{-\infty} P_f(x)dx \geq \int_{-\infty} P_g(x)dx$ for all *x* U'(w) > 0 ; U"(w) < 0
Third order	$\int_{-\infty}(\int_{-\infty} P_f(x)dx)dx \geq \int_{-\infty}(\int_{-\infty} P_g(x)dx)dx$ for some *x* $\int_{-\infty}(\int_{-\infty} P_f(x)dx)dx \geq \int_{-\infty}(\int_{-\infty} P_g(x)dx)dx$ for all *x* U'(*w*) > 0 ; U"(*w*) < 0 ; A'(*w*) < 0

Stochastic Dominance Conditions

First order stochastic dominance by security *f* over security *g* implies that for each potential security payoff x^*, the probability that security g has a smaller payoff $p_g(x < x^*)$ than x^* exceeds (or equals with at least one instance exceeding) the probability that security f will have a smaller payoff than x^*. Thus, for each state of nature x with density (probability) p(*x*), security *f* has at least as high a payoff as does *g* (and in least one state, a higher payoff). The probability that security *g* has a payoff lower than some specified amount exceeds (or equals with at least one instance exceeding) the probability that *f* will have a payoff lower than that amount. Investors preferring more to less will favor the security that exhibits first order stochastic dominance over another. Thus, when U'(*w*)>0, security *f* is preferred to security *g*.

Second order stochastic dominance is concerned with the dispersion of payoffs. Second order stochastic dominance exists when the cumulative distribution function (which is the cumulative-cumulative density function) for security *f* never exceeds the cumulative distribution function for security *g*. In other terms, the cumulative distribution function that *g* has a payoff lower than some specified amount exceeds the cumulative distribution that f will have a payoff lower than that amount. Although this connection might be somewhat confusing, second order stochastic dominance essentially implies that if the probability of payoffs for security g at the lower end of the potential range are exceeded by the probability of payoffs at the higher end of the range for *f*, then *f* exhibits second order stochastic dominance over g. Risk averse investors prefer securities which exhibit second order stochastic dominance.

Consider an example where a risk averse investor who prefers more wealth to less has the opportunity to invest in a security f whose future value is a function of a randomly distributed variable x. The density function for f is given by the following: $p_f(x) = 6(x - x^2)$ for $0 \leq x \leq 1$ and 0 elsewhere. The security f will have a payoff equal to f(x) with probability equal to $p_f(x)$. In addition, the investor has the opportunity to purchase a second security g whose density function is given by $p_g(x) = 12(x^2 - x^3)$ for $0 \leq x \leq 1$ and 0 elsewhere. Density functions for the payoffs for securities f and g are given in the upper graph in Figure 5. Security g will have a payoff equal to g(x) with probability equal to $p_g(x)$. For the sake of simplicity, we shall assume that f(x) = g(x) = x. If the investor is to choose one of the two securities based on first order stochastic dominance criteria, he will first determine cumulative densities (ignoring constants of integration) as follows:

$$\text{(A)} \qquad P_f(x) = \int P_x x d(x) = \int 6\left[x - x^2\right]dx = 6\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]$$

$$\text{(B)}\ \ P_g(x) = \int P_g\, x d(x) = \int 12\left[x^2 - x^3\right] dx = 12\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]$$

These integrals are plotted in the lower graph in Figure 5. Notice that $P_f(x) < P_g(x)$ for all x. This means that security *f* has a higher probability of a smaller payoff than *g* at every potential payoff x. $P_f(x) \geq P_g(x)$ for all x; therefore, security *g* will be preferred to security *f*. First, note that $p_f = p_g = 0$ when $x = 0$ and when $x > 1$. Also note that $P_f = P_g = 1$ when x = 1. However, we can demonstrate with algebra that when $0 < x < 1$, $P_f > P_g$. Thus, the probability that the payoff on security f is less than any constant in the range (0,1) is never less (though it may be greater) than the probability that the payoff on security g will be less than that constant. Thus, security *g* is preferred to security *f*.

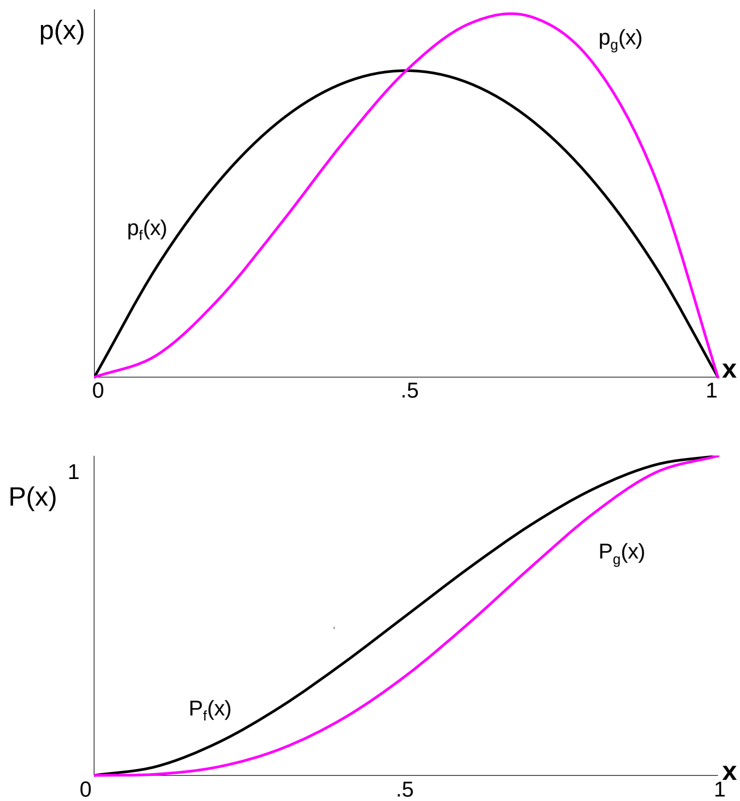

Figure 5: First Order Stochastic Dominance

We can also demonstrate that $\int P_f \geq \int P_g$ for all x; therefore, security g exhibits second order stochastic dominance over security *f*. We integrate Equations (A) and (B) above to obtain:

$$\text{(C)}\qquad P_f(x) = \int \int P_f\, x d(x) = \int 6\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right] = \left[x^3 - \frac{1}{2}x^4\right]$$

$$\text{(D)}\qquad P_g(x) = \int \int P_g\, x d(x) = \int 12\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right] = \left[x^4 - \frac{3}{5}x^5\right]$$

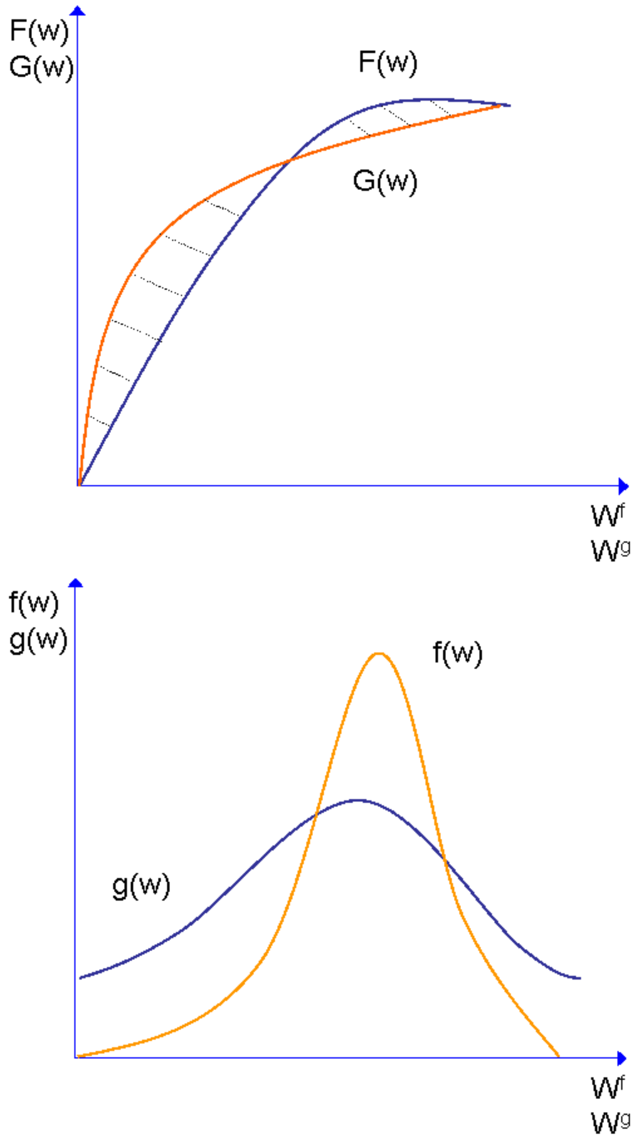

Figure 6: Second Order Stochastic Dominance

Again, note that $\int P_f = \int P_g = 0$ when $x \leq 0$ and $\int P_f = \int P_g = 1$ when $x \geq 1$. However, we can demonstrate with algebra that when $0 < x < 1$, $\int P_f > \int P_g$. Thus, the risk averse investor will prefer security g to security f, even if first order stochastic dominance did not exist.

Whenever first order stochastic dominance exists, investors preferring more to less will choose the asset that exhibits first order dominance. Risk averse investors who prefer more to less will always prefer an asset which exhibits second order stochastic dominance over an alternative asset, regardless of whether first order stochastic dominance exists.

Many types of portfolio selection models make assumptions regarding either the form of probability distribution of returns faced by investors or about the form of investor utility of wealth functions. For example, the Capital Asset Pricing Model assumes either that security returns are normally distributed or that investors have quadratic utility functions. In reality, measurement of investor utility functions is, at best, extremely difficult and the actual probability distribution of security returns is subject to argument. The concept of stochastic dominance provides an alternative to these utility function specific and probability distribution specific portfolio models without excessive reliance on the forms utility and distribution functions. Stochastic dominance may be a useful portfolio selection tool when we are able to make very general assumptions regarding utility and probability functions.

G. The Allais Paradox and the Ellsberg Paradox

<u>The Allais Paradox</u>

The *Allais Paradox* is that risk-averse persons' choices between alternatives seem to vary according to the absolute amounts of potential gains or losses involved in different gambles even when rational choice between gambles should depend only on how the alternatives differ. For example, consider the following example choice of gambles:

Gamble A: .33 probability of receiving 2,500, .66 of receiving 2400 and .01 of receiving 0
Gamble B: 100% probability of receiving 2,400

Kahneman and Tversky found that 82% of their experiment participants preferred Gamble B to Gamble A. However, they offered the same set of participants the following second set of gambles:

Gamble A*: .33 probability of receiving 2,500, .67 of receiving 0
Gamble B*: .34 probability of receiving 2,400 and .66 of receiving 0

In the second part of this experiment, 83% of participants preferred Gamble A* to B*. The same change was made to both gambles in moving from the first to second sets; .66 probability was shifted from Gambles A and B to A* and B* from winnings of 2,400 to zero. The gamble shifts were identical, but many participants reversed their preferences. Yet from the first to second sets of choices, the changes to both gambles were identical; losses of 2,400 were imposed on both gambles from the first set to the second set with probability .34. Since the losses were identical, participants should not have reversed their decisions, but, clearly, the majority did. Kahneman and Tversky surmised that people are risk averse when evaluating positive outcomes (winnings), but risk-seeking when evaluating losses. Hence, people have diminishing utility of wealth functions with respect to winnings, but increasing marginal utility when faced with wealth decreases. Investors seem to exhibit similar reactions to reductions in wealth.

Consider a very simple variation on this problem. One group of subjects was presented with this problem:

1. In addition to whatever you own, you have been given $1,000. You are now asked to choose between:
   A: A sure gain of $500
   B: A 50% change to gain $1,000 and a 50% chance to gain nothing.

A second group of subjects was presented with another problem.

2. In addition to whatever you own, you have been given $2,000. You are now asked to choose between:
   A*: A sure loss of $500

B*: A 50% chance to lose $1,000 and a 50% chance to lose nothing.

In the first group 84% chose A. In the second group 69% chose B*. The two problems are identical in terms of terminal wealth to the subject. However the phrasing of the question causes the problems to invoke different emotional responses. This illustrates the framing versus substance problem.

The Ellsberg Paradox

The *Ellsberg Paradox* concerns inconsistencies in individual decision-making, and has been demonstrated repeatedly in experimental economics.[6] We will consider it in the context as an experiment. Suppose that the subject has an urn that contains 30 red balls and 60 other balls, all of which are either black or yellow. The number of black balls and the number of yellow balls are unknown to the subject, but total of black and yellow balls total number is 60. The balls are well mixed in the urn. Suppose that you are the subject in the experiment given this choice between two gambles:

Gamble A: You receive $100 if you draw a red ball
Gamble B: You receive $100 if you draw a black ball

Most experimental subjects prefer Gamble A. Next, suppose that you are given the choice between these two gambles:

Gamble A*: You receive $100 if you draw a red or yellow ball
Gamble B*: You receive $100 if you draw a black or yellow ball

Most experimental subjects prefer Gamble B*. However, under the von Neumann axioms, you should prefer Gamble B* to Gamble A* if and only if you prefer Gamble A to Gamble B. That is, you should select Gamble A if you believe that drawing a red ball is more likely than drawing a black ball; under this circumstance, you should also prefer Gamble A*. This experiment illustrates individual aversion to ambiguity.[7]

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[6] Daniel Ellsberg, a government policy analyst with substantial experience in Viet Nam during the U.S. war there was more notable for his leaking of the so-called Pentagon papers to the New York Times. This document was compiled by numerous government policy analysts (including himself), exposing decades of failed administrative policy and deceit. Concerned that President Nixon would rather escalate the Viet Nam conflict rather than admit defeat, Ellsberg tried unsuccessfully to present the Pentagon Papers as testimony to Congress. He sent the papers to the *New York Times* instead. He was charged with 12 felonies facing sentences totaling 115 years. The Nixon administration illegally burglarized his psychiatrist's office, wire tapped his phones and attempted to bribe the judge in his case. The administration's illegal activities against Ellsberg were eventually discovered and charges were dropped against Ellsberg. This case ultimately led to 2 of the 3 counts in the impeachment proceedings later filed against President Nixon.

[7] This result was published simultaneously and independently by Daniel Ellsberg and William Fellner and earlier by John Maynard Keynes.

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Exercises

1. Suppose that you are presented with the following choice:

I. In addition to whatever you own, you have been given $1,000. You are now asked to choose between:

A: A sure gain of $500

B: A 50% change to gain $1,000 and a 50% chance to gain nothing.

Which do you prefer: Gamble A or Gamble B? Now, suppose that you are presented with the following choice.

II. In addition to whatever you own, you have been given $2,000. You are now asked to choose between:

A*: A sure loss of $500

B*: A 50% chance to lose $1,000 and a 50% chance to lose nothing.

In experimental studies involving both of these sets of gambles, 84% of study participants chose A over B, however, 69% chose B* over A*. With which of the VNM axioms would the results of this study seem inconsistent?

2. Suppose that an investor with $2 in capital has a logarithmic utility of wealth function: $U = ln(w)$. The investor has the opportunity to buy into the gamble described in the St. Petersburg Paradox. Assume that the investor can borrow without interest and that the gamble payoff is 2^i where i is the number of tosses or outcomes realized before the first head is realized.

a. What is the investor's current utility of wealth level?

b. How much would the investor be willing to pay for the gamble described in the St. Petersburg Paradox?

c. How much would the investor be willing to pay for the gamble described in the St. Petersburg Paradox if his initial wealth level were $1,000 rather than $2?

d. What would be your answer to part b if the gamble payoff were to change to 2^{2i-1} where i is the number of tosses or outcomes realized before the first head is realized?

3. A car with a replacement value of $20,000 can be insured against a total loss with an insurance policy sold for a premium of $1,200. The insurance company selling the policy and the consumer purchasing the policy agree that there is a 5% probability that the car will be destroyed.

a. What is the actuarial (fair or expected) value of the policy?
b. If the insurance maintains a large, well-diversified portfolio of such policies, what is its expected profit from the sale of this policy?
c. What is the expected profit (or gain or loss) to the consumer from the purchase of this policy?
d. Under what circumstances is the sale of this policy a rational transaction for the risk-neutral insurance company?
e. Under what circumstances is the purchase of this policy a rational transaction for the consumer?

4. Define an investor's utility (U) as the following function of his wealth level (w): $U = 1000w - .01w^2$. This investor currently has $10,000. Answer the following:

a. What is his current utility level?

b. Find the utility level he would associate with 12,000.
c. Use a Taylor series second order approximation to estimate the investor's utility level after his wealth level is increased by \$2,000 from its current level of \$10,000.
d. What is the investor’s current (w = 10,000) absolute risk aversion coefficient?
e. What is the investor’s current (w = 10,000) relative risk aversion coefficient?
f. Suppose that the investor’s wealth level were to increase to 12,000. What would be the investor’s new absolute and relative risk aversion coefficients?
g. How might you interpret your answers to parts e and f? Do the differences between your answers seem inconsistent with what might actually be observed for investors?

5. Suppose that an investor with \$20,000 in capital has a logarithmic utility of wealth function: $U = ln(W)$.
a. Assuming a small risk (Arrow Pratt), what would be his Coefficient of Absolute Risk Aversion (ARA)?
b. What would the investor be willing to pay to insure the risk associated with a gamble that would pay \$10 with probability 50% or pay -\$10 with probability 50%?
c. Assuming a small risk (Arrow Pratt), what would be his Coefficient of Relative Risk Aversion (RRA)?
d. How does the investor’s Coefficient of Relative Risk Aversion (RRA) change as his wealth level changes? What does this imply about his propensity to assume risk as he becomes wealthier?

6. Suppose that a consumer with a utility function $U = aW – bW^2 = 1000W - .01W^2$ has a wealth level equal to 10,000 if no loss occurs. This wealth includes 6,000 in cash and a car worth 4,000 if no crash occurs. The consumer has 6,000 in cash along with a gamble that will incur a loss with probability p = .5; the gamble will lose 4,000 with probability p = .5 and nothing otherwise. The consumer can purchase insurance on any fractional amount $0 \le \alpha \le 1$of this gamble with a premium of $\pi = pcL$. For example, if the investor insures fraction $\alpha = .8$ or 4,000 of this gamble, the most that she can lose is 1,000, her coinsurance amount. Each unit of insurance costs pc = .505, and up to 4,000 units can be purchased. Thus, insurance on 4,000 of the loss costs 2,020.
a. Is the insurance premium actuarially fair?
b. How much insurance (α) should the investor purchase to maximize her utility?
c. What will be the premium on this optimal policy?
d. What is the expected casualty loss to the insurance company?
e. What is the expected profit on the policy to the insurance company?
f. What is the optimal level of coinsurance ($1- \alpha$) on this gamble for the consumer if c = 1.02?

7. Suppose that an investor has the following quadratic utility function with respect to consumption c_T:

$$U(c_T) = ac_T – b\, c_T{}^2 \quad \text{for } a, b > 0;\ a > 2bc_T$$

a. Write a function to characterize the investor's expected utility of terminal consumption.
b. Write a function to characterize the investor's risk aversion coefficient (ARA).
c. Suppose that the investor seeks to invest some level S_0 of his wealth W_0 in risky assets with normally distributed random return $\tilde{r}$ and the remainder to be invested in the risk free security with return r_f. Write a function to characterize the investor's expected utility of terminal

consumption.

d. Write a function that calculates the optimal holdings S_0 in the risky asset.

e. Describe how holdings in risky assets will vary with respect to returns on the risky asset, the riskless asset return, the risk of the risky asset and initial wealth.

8. In our discussion of the Allais Paradox, we considered the following choice of gambles:

Gamble A: .33 probability of receiving 2,500, .66 of receiving 2400 and .01 of receiving 0
Gamble B: 100% probability of receiving 2,400

and

Gamble A*: .33 probability of receiving 2,500, .67 of receiving 0
Gamble B*: .34 probability of receiving 2,400 and .66 of receiving 0

a. Demonstrate that if an investor is indifferent between Gambles A and B, he must be indifferent to A* and B* in order to fulfill the *Strong Independence axiom* identified by von Neumann and Morgenstern.

b. Suppose that the investor's utility of wealth function is given to be $U_w = ln(1+w)$. Calculate expected utilities of Gambles A, B, A* and B*.

c. Based on expected utilities, which gamble in each pair is preferred?

9. Consider the following listing state-contingent payoffs for Investments A, B and C:

Investment	State 1	State 2	State 3	State 4
A	12	13	14	14
B	11	12	14	16
C	10	14	14	15

Assume that each potential state is equally likely. List all (if any) cases of stochastic dominance on the first, second and third orders.

10. Assume that the density function p_f for a randomly distributed variable $\{p_f(x) = P(x)\}$ is given by the following: $p_f(x) = 3x^2$ for $0 \leq x \leq 1$ and 0 elsewhere. A second density function p_g for a randomly distributed variable $\{p_g(x) = P(x)\}$ is given by the following: $p_g(x) = (2x^3+x)$ for $0 \leq x \leq 1$ and 0 elsewhere.

a. Find $P_f(x)$ and $P_g(x)$.
b. Demonstrate whether there exist conditions of First Order Stochastic Dominance.
c. Demonstrate whether there exist conditions of Second Order Stochastic Dominance.

11. Suppose that an investor has the opportunity (and funding ability) to pay \$100,000 for a 50% chance to win \$300,000 and a 50% chance of winning nothing.

a. What is the expected value of the gamble?

b. What is the standard deviation of payoffs for this gamble?

c. Suppose that you have the opportunity (and funding ability) to repeat participation in this gamble for a total of 5 gambles. Each wager's outcome is independent of the outcomes of all other wagers (the correlation coefficient between wager payoffs is zero). What is the expected value of this set of 5 wagers?

d. What is the standard deviation for this set of 5 wagers?

e. Which set of wagers has a higher expected payoff, that described in parts a and b of this question or that described in parts c and d of this question?

f. Which set of wagers has a lower risk as measured by standard deviation, that described in parts a and b of this question or that described in parts c and d of this question?

g. Which set of wagers seems to be preferable based on your answers to parts a through f, the single wager or the set of 5 wagers?

h. Devise an argument that if an individual finds the gamble described in parts a and b unacceptable, he will also find the gambles described in parts c and d unacceptable.

Solutions

1. The two sets of choices are identical in terms of terminal wealth to the subject. However the phrasing of the question causes the problems to invoke different emotional responses; that is, the framing is different. This leads to the following framing versus substance problem, which is a violation of the von Neumann-Morgenstern independence axiom.

2.a. $E[U_{\text{w, No Gamble}}] = ln(2) = .693147$

b. Solve the following for G, where G is the cost of the gamble and x is its winnings:
$E[U_{\text{w, With gamble}}] = \sum_{i=1}^{\infty} p_i\, U(w + x_i - G) = \sum_{i=1}^{\infty}\left[.5^i ln\left(2 + 2^i - G\right)\right] = .693147$
G = 3.34757
Note: A spreadsheet may be useful to solve this infinite series. The value G is the payment for the gamble, the initial wealth level is 2, winnings are 2^i where i is the number of tosses before the first head. The value of G is obtained by iteration. The following is the first 13 rows of spreadsheet calculations for this problem:

	$\underline{p^i}$	$\underline{(2\text{-}G+2^i)}$	$\underline{(p^i)(2\text{-}G+2^i)}$	$\underline{SUM(p^i)(2\text{-}G+2^i)}$
1	0.5	-0.42705	-0.213525713	-0.213525713
2	0.25	0.975476	0.243869050	0.030343338
3	0.125	1.894982	0.236872775	0.267216113
4	0.0625	2.684606	0.167787887	0.435004000
5	0.03125	3.422712	0.106959748	0.541963748
6	0.015625	4.137602	0.064650039	0.606613787
7	0.0078125	4.841447	0.037823801	0.644437588
8	0.00390625	5.539900	0.021640233	0.666077821
9	0.00195313	6.235689	0.012179080	0.678256901
10	0.00097656	6.930155	0.006767729	0.685024630
11	0.00048828	7.623961	0.003722637	0.688747268
12	0.00024414	8.317437	0.002030624	0.690777892
13	0.00012207	9.010749	0.001099945	0.691877837

A trial value G is entered elsewhere in the spreadsheet and this cell is referenced for all other cells where G is used. The value for G is iterated until the sum is sufficiently close to the natural log of 2. In this table, the value 3.34757 is used, where this value was obtained by trial and error in an effort to obtain .693147 (or some sufficiently close value) for the sum in the 13th row.

c. First, find the utility of \$1000: $ln(1000) = 6.907755$
Now, solve the following for G, where G is the cost of the gamble:
$E[U_{\text{w, With gamble}}] = \sum_{i=1}^{\infty} p_i\, U(w + x_i - G) = \sum_{i=1}^{\infty}\left[.5^i ln\left(2 + 2^i - G\right)\right] = 6.907755$
G = 10.954
Note: See the following table excerpted from a spreadsheet used to solve the infinite series, iterating for G. The key column, "Contribution to Utility" equals Probability * *ln*(1000+ "Gamble Payoff" – G). The column is then summed such that the sum equals the utility of \$1,000:

Toss	Probability	Gamble Payoff	Log of Post-Gamble Wealth	Contribution to Utility	Contribution to Wealth
1	0.5	2	6.898761	3.449380476	1

2	0.25	4	6.900777	1.725194247	1
3	0.125	8	6.904797	0.863099613	1
4	0.0625	16	6.912789	0.432049287	1
5	0.03125	32	6.928583	0.216518215	1
6	0.015625	64	6.959442	0.108741284	1
7	0.0078125	128	7.018443	0.054831586	1
8	0.00390625	256	7.126928	0.027839562	1
9	0.00195313	512	7.313917	0.014284995	1
10	0.00097656	1024	7.607404	0.007429106	1
11	0.00048828	2048	8.018641	0.003915352	1
12	0.00024414	4096	8.534059	0.002083511	1
13	0.00012207	8192	9.124896	0.001113879	1
14	6.1035E-05	16384	9.762675	0.000595866	1
15	3.0518E-05	32768	10.42694	0.000318205	1
16	1.5259E-05	65536	11.10533	0.000169454	1
17	7.6294E-06	131072	11.79102	8.99583E-05	1

d. Solve the following for G, where G is the cost of the gamble:

$E[U_{\text{w, With gamble}}] = \sum_{i=1}^{\infty} p_i\, U(w + x_i - G) = \sum_{i=1}^{\infty}\left[.5^i ln\left(2 + 2^{2i-1} - G\right)\right]$

$= (2 + 1 - G) + (2 + 8 - G) + (2 + 32 - G) + (2 + 64 - G) + \cdots$

For any finite value of G, expected utility must equal ∞. Thus G = ∞, and an investor would be willing to pay any finite sum for this gamble. That is, this illustration shows how an investor, with diminishing marginal utility of wealth (log utility function) and risk aversion would still be willing to pay an infinite sum of money for a gamble. This example might be referred to as a "Super St. Petersburg Paradox."

3.a. Based on expected value, the actuarial value of this policy is .05 · \$20,000 = \$1,000.

b. \$1,200 – 1,000 = \$200

c. \$1,000 – 1,200 = -\$200; \$200 expected loss to the consumer

d. The sale is a rational transaction if the insurance company intends to increase its wealth (more is preferred to less)

e. The purchase is a rational transaction to the consumer if she is sufficiently risk-averse.

4. Answers are as follows:

a. $U_{10,000}$ = 10,000,000 - .01 x 100,000,000 = 9,000,000

b. $U_{12,000}$ = 12,000,000 - .01 x 144,000,000 = 10,560,000

c. $U_{10,000 + 2,000}$ = 9,000,000 + (1000 - .02x10,000) x 2000 + (1/2) x (-.02) x 2000^2 = 10,560,000

d. ARA = -U''(w)/U'(w) = -(-.02)/[1,000 - (.02 · 10,000)] = .02/800 = .00025

e. RRA = -wU''(w)/U'(w) = -10,000 · -.02/[1,000 - (.02 · 10,000)] = 200/800 = .25

f. ARA = -U''(w)/U'(w) = -(-.02)/[1,000 - (.02 · 12,000)] = .02/760 = .00026315789

RRA = -wU''(w)/U'(w) = -12,000 · -.02/[1,000 - (.02 · 12,000)] = 240/760 = .315789474

g. Both absolute and relative risk aversion coefficients increase as wealth increases. This suggests that investors become more risk averse, and are less willing to assume risks of a given monetary amount (or, proportional amount) as they become wealthier. While this result is typical for quadratic utility functions, it is inconsistent with empirical observations. Thus, this tendency does weaken the applicability of the quadratic utility function.

5.a. Based on the standard Arrow-Pratt risk aversion model, we first calculate the first derivative of utility with respect to wealth: dU/dW = d*ln*(*W*)/dW = 1/W. Next, we calculate the second derivative of utility with respect to wealth: $d^2U/dW^2 = d(1/W)/dW = -1/W^2$. Now, we calculate ARA as follows: ARA = -U''(W)/U'(W) = $-(-1/W^2)/(1/W)$ = 1/W = 1/20,000

b. $\sigma^2_z = 0.5\times(20{,}010 - 20{,}000)^2 + 0.5\times(19{,}090 - 20{,}000)^2 = 100$
$\pi = -(1/2)*100\times(-1/20{,}000) = .0025$

c. RRA = -wU''(w)/U'(w) = $-w(-1/w^2)/(1/w)$ = w/w = 1

d. ∂RRA/∂w = 0; Relative risk aversion is constant for a logarithmic utility function. This means that an investor's propensity to assume proportional risks does not change as his wealth changes. This log utility function is a constant relative risk aversion utility function.

6.a. No; c = 1.01 > 1

b. The consumer with this quadratic utility function will seek to maximize her expected utility:

$E[U] = (1-p)\cdot[a\cdot(W-cpL\alpha) - b\cdot(W-cpL\alpha)^2] + p\cdot[a\cdot(W-L-cpL\alpha + L\alpha) - b\cdot(W-L-cpL\alpha + L\alpha)^2]$

The derivative of E[U] with respect to α is expressed as follows:

$\frac{\partial E[U]}{\partial \alpha} = (1-p)\cdot[-cpLa - 2b\alpha(cpL)^2 + 2bcpLW + p\cdot[a(L-cpL) - 2b(L-cpL)^2\alpha - 2b(L-cpL)(W-L)] = 0$

where the following inputs are applied:

a =	1000	b =	0.01
w =	10000	L =	4000
p =	0.5	α =	0.789921
c =	1.01		

Substituting in numerical values and simplifying, we find that the optimal insurance level is α = .789921. The consumer will purchase insurance to insure a loss of 3159.684.

c. π = αpcL = 1595.6404

d. E[L] = .5 × 3159.6404 = 1579.8202

e. E[Profit] = 1595.6404 – 1579.8202 = 15.8202

f. Solve the following for α:

$dE[U]/d\alpha = (1-.5)\cdot[-1.02\cdot 4{,}000\cdot\alpha - 2\cdot.01\cdot\alpha(1.02\cdot.5\cdot4{,}000)^2 + 2\cdot.01\cdot1.02\cdot.5\cdot4{,}000\cdot10{,}000$
$+ .5\cdot[\alpha\cdot(4{,}000-1.02\cdot.5\cdot4{,}000) - 2\cdot.01\cdot(4{,}000-1.02\cdot.5\cdot4{,}000)^2\cdot\alpha$
$-2\cdot.01\cdot(4{,}000-1.02\cdot.5\cdot4{,}000)\cdot(10{,}000-4{,}000)] = 0$; α = 0.579768094 by substitution;
(1- α) = .42024

7.a. $E[U(c_T)] = aE[c_T] - bE[c_T^2]$

b. $ARA = -\frac{U''(c_T)}{U'(c_T)} = \frac{2b}{a-2bc_T}$

c. First, we write terminal consumption as follows:

$$c_T = S_0\tilde{r} + (W_0 - S_0)r_f = W_0r_f + S_0(\tilde{r} - r_f)$$

Utility of terminal consumption is written:

$$E[U(c_T)] = E[aE[c_T] - bE[c_T^2]] = aE[W_0r_f + S_0(\tilde{r} - r_f)] - bE[W_0r_f + S_0(\tilde{r} - r_f)]^2$$

d. $E[U(c_T)] = aE[W_0r_f + S_0(\tilde{r} - r_f)] - b[W_0r_f]^2 - bS_0^2E[\tilde{r} - r_f]^2 - 2bW_0r_fS_0E[\tilde{r} - r_f]$

The first order condition for expected utility maximization is as follows:

$$\frac{\partial E[U(c_T)]}{\partial S_0} = aE\left[\tilde{r} - r_f\right] - 2bS_0E\left[\tilde{r} - r_f\right]^2 - 2bW_0r_fE\left[\tilde{r} - r_f\right] = 0$$

$$S_0 = \frac{aE\left[\tilde{r} - r_f\right] - 2bW_0r_fE\left[\tilde{r} - r_f\right]}{2bE\left[\tilde{r} - r_f\right]^2} = \frac{aE\left[\tilde{r} - r_f\right] - 2bW_0r_fE\left[\tilde{r} - r_f\right]}{2b\sigma^2}$$

e. Holdings in the risky asset with increase with risky asset returns. Holdings in the risky asset will decrease as the riskless asset return, the risk of the risky asset and initial wealth increase (assuming the usual restrictions on *a* and *b*).

8.a. First, since a .66 probability of a $2,400 payout is being shifted to 0 from A and B to A* and B*, we will rewrite the statement of gamble payoffs as follows:

Gamble A: .33 probability of receiving 2,500, .66 of receiving $2,400 and .01 of receiving 0
Gamble B: .34 probability of receiving 2,400 and .66 of receiving $2,400
and
Gamble A*: .33 probability of receiving 2,500, .01 of receiving 0 and .66 of receiving 0
Gamble B*: .34 probability of receiving 2,400 and .66 of receiving 0

The investor is indifferent between Gambles A and B. Recall that the Strong Independence axiom states that if $x_j \succ x_k$, then for any $\alpha \in [0,1]$, $\alpha x_i + (1-\alpha)x_k \sim \alpha x_j + (1-\alpha)x_k$. This Strong Independence axiom implies that for any $\alpha \in [0,1]$:

α(.33 prob. of receiving 2,500 and .01 of receiving 0) + $(1-\alpha)$(.66 prob. of receiving 2,400)
~ α(.34 prob. of receiving 2,400) + $(1-\alpha)$(.66 prob. of receiving 2,400),

which implies that:

(.33 prob. of receiving 2,500 and .01 of receiving 0) ~ (.34 prob. of receiving 2,400)

The same decomposition for Gambles A* and B* results in:
Gamble A*: α(.33 prob. of receiving 2,500, .01 of receiving 0) + $(1-\alpha)$(.66 prob. of receiving 0)
Gamble B*: α(.34 prob. of receiving 2,400) + $(1-\alpha)$(.66 prob. of receiving 0),
Which, by the same strong independence axiom, reduces to a comparison between:
Gamble A*: (.33 prob. of receiving 2,500 and .01 of receiving 0)
Gamble B*: (.34 prob. of receiving 2,400)
We know from our statement above concerning Gambles A and B
(.33 prob. of receiving 2,500 and .01 of receiving 0) ~ (.34 prob. of receiving 2,400)
that the investor must be indifferent between Gambles A* and B*.

b. First, we calculate the utilities of the three potential wealth levels:

U(2500) = 7.824; U(2400) = 7.783; U(0) = $-\infty$

Next, we calculate the expected utilities of the gambles:

U(A) = .33*7.824 + .66*7.783 * .01*0 = 7.71927
U(B) = 7.783
U(A*) = .33*7.824 + .67*0 = 2.582067
U(B*) = .34*7.824 + .66*0 = 2.660312

c. The expected utilities of B and B* exceed those of A and A*.

9. Because all outcomes have equal associated probabilities, we can rank-order the payoffs and look for situations involving stochastic dominance. We will first rank order cash flows for the investments and seek conditions of first order stochastic dominance:

A 12 13 14 14

B 11 12 14 16
C 10 14 14 15

First order stochastic does not exist. There is no investment whose cash flow is less than that of another investment in each and every case. Thus, when more is preferred to less, and there are no additional preferences that we can use to rank investments, we cannot select among the investments. Next, to seek conditions of second order stochastic dominance, we sum the investment cash flows, starting with the worst outcomes at each successive improved outcome:

A 12 25 39 53
B 11 23 37 53
C 10 24 38 53
A SSD B

Second order stochastic between A and B exists here because B never has a larger sum of cash flows than A, and sometimes have smaller cash flows. Thus, when more is preferred to less, and safety is preferred to risk, A will be preferred to B. Next, to seek conditions of third order stochastic dominance, we sum the sums of investment cash flows, starting with the worst outcomes at each successive improved outcome:

A 12 37 76 129
B 11 34 71 124
C 10 34 72 125
A TSD B; A TSD C

Whenever second order stochastic dominance exists between a pair of investments, third order stochastic exists between that pair. Thus, A stochastically dominates B in the third order. In addition, A stochastically dominates C in the third order. Thus, when more is preferred to less, safety is preferred to risk, and positive skewness is preferred to negative skewness, A will dominate, B and C in the third order, and will be preferred to them as well.

10. First, we integrate the density functions (ignoring constants of integration):

a. $P_f(x) = \int 3x^2 = x^3$ for 0<x<1

$$P_g(x) = \int (2x^3 + x) = \frac{1}{2}x^4 + \frac{1}{2}x^2 \quad \text{for } 0<x$$

b. $\frac{1}{2}x^4 + \frac{1}{2}x^2 > x^3 \; for \;\; 0 \le x \le 1$

Thus, security f exhibits first order stochastic dominance over security g.

c. $\int \left[\frac{1}{2}x^4 + \frac{1}{2}x^2\right] \ge \int x^3$

$$\int \left[\frac{1}{10}x^5 + \frac{1}{6}x^3\right] \ge \frac{1}{4}x^4 \; for \; 0 \le x \le 1$$

Thus, security f exhibits second order stochastic dominance over security g. Note that first order stochastic dominance always implies second order stochastic dominance.

11.a. .5(300,000 – 100,000) + .5(0 – 100,000) = 50,000
b. $[.5(200{,}000-50{,}000)^2 + .5(-100{,}000-50{,}000)^2]^{.5} = 150{,}000$
c. 5*50,000 = 250,000

Alternatively, in 5 wagers, with 120 (5!) possible win/loss scenarios, there is 1 scenario in which

5 consecutive losses totaling 500,000 occur with a probability of $.5^5$, 5 scenarios in which 4 losses and 1 win occur with a net loss of 200,000 and a probability of $5*.5^5$ and so on:

$$1*.5^5*(-500000)+5*.5^5*(-200000)+10*.5^5*100000+10*.5^5*400000+5*.5^5*700000 +1*.5^5*1000000 = 250,000$$

d. $(1*0.5^5*(-500000-250000)^2 + 5*0.5^5*(-200000-250000)^2 + 10*0.5^5*(100000-250000)^2 + 10*0.5^5*(400000-250000)^2 + 5*0.5^5*(700000-250000)^2 + 1*0.5^5*(1000000-250000)\text{^}2)^{.5} = 335,410.2$

Alternatively, since payoffs from each of the 5 wagers are independent of one another, this standard deviation of 5 gambles can be computed as follows:

$$(5*150,000^2)^{.5} = 335,410.2$$

e. Obviously, the set of 5 wagers described in parts c and d has the higher expected value.

f. Obviously, the set of 5 wagers described in parts c and d has the higher standard deviation. However, notice that this higher standard deviation is less than 5 times the individual wager standard deviation.

g. The answer to this depends on how you evaluate the wagers and your own individual preferences. However, the reward to risk ratio for the single wager 50,000/150,000 = 3 is much greater than the reward to risk ratio of the set of 5 gambles 250,000/335.410.2 = .746. This suggests that as the number of gambles increases, the reward to risk ratio will also increase due to diversification. However, consider the response to the next part of this question.

h. Suppose that you find the individual wager described in parts a and b unacceptable. Then, if after having wagered 4 times (the first 4 of 5 wagers described in parts c and d), you have the opportunity to wager a 5th time, you should decline, since you find any single wager of this type to be unacceptable. Then, by the same logic, after having wagered 3 times, you would find the 4th wager unacceptable, and so on. Thus, any person finding the single wager described in parts a and b to be unacceptable should also find the set of 5 wagers to be unacceptable. Hence, diversification over time or over a series of sequential gambles would not mitigate risk. This is the substance of the Paul Samuelson "Law of Large Numbers" fallacy (Samuelson [1963]).