IV. MODERN PORTFOLIO THEORY

1. Elementary Portfolio Arithmetic

A portfolio is simply a collection of investments held by an investor. It may be reasonable to be concerned with the performance of individual securities only to the extent that their performance affects overall portfolio performance. Thus, the performance of the portfolio is of primary importance. The return of an investor's portfolio is simply a weighted average of the returns of the individual securities that within it. The expected return of a portfolio may be calculated either as a function of potential portfolio returns and their associated probabilities (as computed in earlier sections) or as a simple weighted average of the expected individual security returns. Generally, the portfolio variance or standard deviation of returns will be less than a weighted average of the individual security variances or standard deviations.

Portfolio Return

The expected return of a portfolio may be calculated using Equation 1 where the subscript p designates the portfolio and the subscript j designates a particular outcome out of m potential outcomes:

(1) $$E[R_p] = \sum_{j=1}^{m} R_{pj} \cdot P_j$$

For many portfolio management applications, it is useful to express portfolio return as a function of the returns of the individual securities that comprise the portfolio:

(2) $$E[R_p] = \sum_{i=1}^{n} w_i \cdot E[R_i]$$

The subscript i designates a particular security, and weights w_i are the portfolio proportions. That is, a security weight w_i specifies how much money is invested in security i relative to the total amount invested in the entire portfolio. For example, w_i is:

$$w_1 = \frac{\text{\$ invested in security 1}}{\text{Total \$ invested in the portfolio}}$$

Thus, portfolio return is simply a weighted average of individual security returns.

Portfolio Risk

We can also define portfolio return variance as a function of potential portfolio returns and associated probabilities:

(3) $$\sigma_p{}^2 = \sum_{i=1}^{n} (R_{pi} - E[R_p])^2 P_i$$

It is important to note that the variance of portfolio returns usually is not simply a weighted average of individual security variances. In fact, in some instances, we can combine a series of highly risky assets into a relatively safe portfolio. The risk of a portfolio in terms of variance of returns can be determined by solving the following double summation:

(4) $$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_i \cdot \sigma_j \cdot \rho_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{ij}$$

When a portfolio is comprised of only two securities, its variance can be determined by Equation

(5):

(5) $$\sigma_p^2 = (w_1^2\ \sigma_1^2) + (w_2^2\ \sigma_2^2) + 2(w_1\ w_2 \sigma_1 \sigma_2 \rho_{1,2})$$

Larger portfolios require the use of Equation 4 or a variant of Equation 5 accounting for all products of security weights and standard deviations squared and all possible combinations of pair-wise security covariances and weight products.

The implication of the covariance terms in Equations 4 through 5 is that security risk can be diversified away by combining individual securities into portfolios. Thus, the old stock market adage "Don't put all your eggs in one basket" really can be validated mathematically. Spreading investments across a variety of securities does result in portfolio risk that is lower than the weighted average risks of the individual securities. This diversification is most effective when the returns of the individual securities are at least somewhat unrelated; that is, lower covariances $\sigma_{i,j}$ result in lower portfolio risk. Similarly, the reduction of portfolio risk is dependent on the correlation coefficient of returns σ_{ij} between securities included in the portfolio. Since the covariance between security returns $\sigma_{i,j}$ equals the product $\sigma_i \sigma_j \rho_{i,j}$, covariance will reflect the correlation coefficient. Thus, the lower the correlation coefficients between these securities, the lower will be the resultant portfolio risk. In fact, as long as ρ_{ij} is less than one, which realistically is always the case between non-identical securities, some reduction in risk can be realized from diversification.

A Simple Derivation of Portfolio Variance

To derive the variance of portfolio *p* as a function of security variances, covariances and weights as in Equation 4, we begin with our standard variance expression as a function of m potential portfolio return outcomes j and associated probabilities as in Equation 3:

(3) $$\sigma_p{}^2 = \sum_{i=1}^{n} (R_{pi} - E[R_p])^2 P_i$$

For the sake of simplicity, let the number of securities n in our portfolio equal two. From our portfolio return expression, we may compute portfolio variance as follows:

$$\sigma_p{}^2 = \sum_{i=1}^{n} \left(w_1 R_{1i} + w_2 R_{2i} - w_1 E[R_1] - w_2 E[R_2]\right)^2 P_i$$

(A)

Next, we complete the square for Equation A and combine terms multiplied by the two weights to obtain:

(B) $$\sigma_p{}^2 = \sum_{i=1}^{n} [w_1^2 (R_{1i} - E[R_1])^2 P_i + w_2^2 (R_{2i} - E[R_2])^2 P_i + 2 w_1\, w_2 (R_{1i} - E[R_1])(R_{2i} - E[R_2])] P_i$$

Next, we bring the summation term inside the brackets:

(C) $$\sigma_p{}^2 = w_1^2 \sum_{i=1}^{n} [(R_{1i} - E[R_1])]^2 P_i + \sum_{i=1}^{n} [w_2^2 (R_{2i} - E[R_2])^2] P_i + 2 w_1\, w_2 \sum_{i=1}^{n} [(R_{1i} - E[R_1])(R_{2i} - E[R_2])] P_i$$

We complete our derivation by noting our definitions for variances and covariances as follows:

(5) $$\sigma_p^2 = (w_1^2\ \sigma_1^2) + (w_2^2\ \sigma_2^2) + 2(w_1\ w_2 \sigma_{1,2})$$

which is a special case of Equation 4. Similar derivations can be performed for portfolios comprising more than two securities.

Simple Numerical Illustration: Two Security Portfolio

For many portfolio management applications, it is useful to express portfolio return as a function of the returns of the individual securities that comprise the portfolio. This is often because we want to know how a particular security will affect the return and risk of our overall holdings or portfolio. For example, consider a portfolio made up of two securities, one and two. The expected return of security one is 10% and the expected return of security two is 20%. If forty percent of the dollar value of the portfolio is invested in security one (that is, $[w_1] = .40$), and the remainder is invested in security two ($[w_2] = .60$), the expected return of the portfolio may be determined by Equation (2):

$$(2) \qquad E[R_p] = \sum_{i=1}^{n} w_i \cdot E[R_i]$$

$$E[R_p] = (.4 \cdot .10) + (.6 \cdot .20) = .16$$

If the standard deviation of returns on securities one and two were .20 and .30, respectively, and the correlation coefficient $\rho_{i,j}$ between returns on the two securities were .5, the resulting standard deviation of the portfolio would be .23, the square root of its .0504 variance level:

$$(4)\,\sigma_p^2 = (.4 \cdot .4 \;\; .2 \cdot .2 \cdot 1) + (.4 \cdot .6 \cdot .2 \cdot .3 \cdot .5) \; + (.6 \cdot .4 \cdot .3 \cdot .2 \cdot .5) \; + \; (.6 \cdot .6 \cdot .3 \cdot .3 \cdot 1) = .0532\,; \sigma_p = .23$$

Diversification and Security Return Correlations

Diversification is simply holding multiple assets whose returns are not perfectly correlated. There are two ways to reduce risk through portfolio diversification:

- Select securities with lower pairwise correlations between returns (Pick unlike securities).
- Increase the number of securities in the portfolio.

However, this list oversimplifies diversification; a bit of focus is needed on appropriate security combinations or weights. Nevertheless, we will focus here on selecting securities that fit our portfolio as we begin our preamble to the efficient set of portfolios.

In our first example, the weighted average of the standard deviation of returns of the two securities one and two is 26%, yet the standard deviation of returns of the portfolio they combine to make is only 23%. Clearly, some risk has been diversified away by combining the two securities into the portfolio. In fact, the risk of a portfolio will almost always be lower than the weighted average of the standard deviations of the securities that comprise that portfolio.

For a more extreme example of the benefits of diversification, consider two securities, three and four, whose potential return outcomes are perfectly inversely related. Data relevant to these securities is listed in Table (1). If outcome one occurs, security three will realize a return of 30%, and security four will realize a 10% return level. If outcome two is realized, both securities will attain returns of 20%. If outcome three is realized, securities three and four will attain return levels of 10% and 30%, respectively. If each outcome is equally likely to occur ($[P_i]$ is .333 for all outcomes), the expected return level of each security is 20%; the standard deviation of returns for each security is .08165. The expected return of a portfolio combining the two securities is 20% if each security has equal portfolio weight ($[w_3] = [w_4] = .5$), yet the standard deviation of portfolio returns is zero. Thus, two relatively risky securities have been combined into a portfolio that is virtually risk-free.

TABLE 1: Portfolio return with perfectly inversely correlated securities. $W_3 = W_4 = 0.5$

I	R_{3i}	R_{4i}	R_{pi}	P_i
1	.30	.10	.20	.333
2	.20	.20	.20	.333
3	.10	.30	.20	.333

Notice in the previous paragraph that we first combined securities three and four into a portfolio and then found that portfolio's return given each outcome. The portfolio's return is 20% regardless of the outcome; thus, it is risk free. The same result could have been obtained by finding the variances of securities three and four, the correlation coefficient between their returns, then solving for portfolio variance with Equation (6) as in Table (2).

TABLE 2: Portfolio return with perfectly inversely correlated securities.

Given:

$\overline{R}_3 = 0.20$ $\quad\quad$ $\overline{R}_4 = 0.20$

σ_3=0.08165 $\quad$ σ_4=0.08165 $\quad$ w_3=0.50 $\quad$ w_4=0.50 $\quad$ $\rho_{3,4} = -1$

Then: $\overline{R}_p = w_3\overline{R}_3 + w_4\overline{R}_4 = (0.5 \times 0.20) + (0.5 \times 0.20) = 0.20$

$$\sigma_p = \sqrt{w_3^2\sigma_3^2 + w_4^2\sigma_4^2 + 2w_3w_4\sigma_3\sigma_4\rho_{3,4}}$$

$$\sigma_p = \sqrt{0.5^2 \times .0066667 + 0.5^2 \times 0.0066667 \ + 2 \times 0.5 \times 0.5 \times 0.08165 \times 0.08165 \times (-1)}$$

$$\sigma_p = \sqrt{0.0016667 + 0.0016667 - 0.003333} = \sqrt{0} = 0$$

The implication of the two examples provided in this chapter is that security risk can be diversified away by combining the individual securities into portfolios. Thus, the old stock market adage "Don't put all your eggs in one basket" really can be validated mathematically. Spreading investments across a variety of securities does result in portfolio risk that is lower than the weighted average risks of the individual securities. This diversification is most effective when the returns of the individual securities are at least somewhat unrelated; or better still, inversely related as were securities three and four in the previous example. For example, returns on a retail food company stock and on a furniture company stock are not likely to be perfectly positively correlated; therefore, including both of them in a portfolio may result in a reduction of portfolio risk. From a mathematical perspective, the reduction of portfolio risk is dependent on the correlation coefficient of returns ($\rho_{i,j}$) between securities included in the portfolio. Thus, the lower the correlation coefficients between these securities, the lower will be the resultant portfolio risk. In fact, as long as (ρ_{ij}) is less than one, which, realistically is always the case, some reduction in risk can be realized from diversification.

Consider Figure (1). The correlation coefficient between returns of securities C and D is one. The standard deviation of returns of any portfolio combining these two securities is a weighted average of the returns of the two securities' standard deviations. Diversification here yields no benefits. In Figure (2), the correlation coefficient between returns on Securities A and B is .5. Portfolios combining these two securities will have standard deviations less than the weighted average of the standard deviations of the two securities. Given this lower correlation coefficient, which is more representative of "real world" correlations, there are clear benefits to diversification. In fact, we can see in Figures (3) and (4) that decreases in correlation coefficients result in increased diversification benefits. Lower correlation coefficients result in lower risk levels at all levels of expected return. Thus, an investor may benefit by constructing his portfolio of securities with low correlation coefficients.

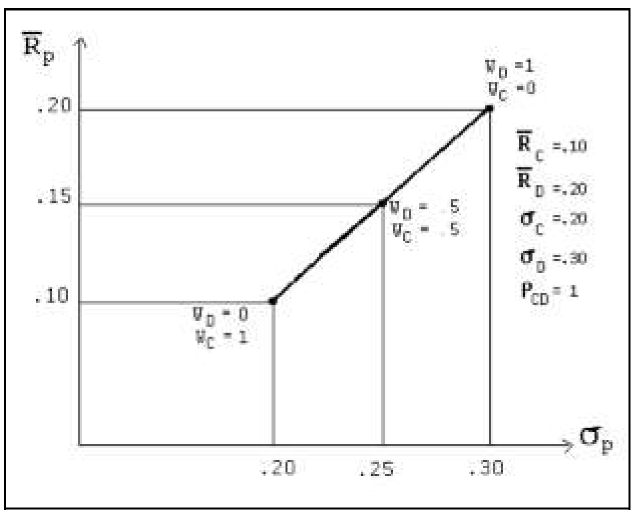


Figure 1: Relationship between portfolio return and risk when ρ_{CD}=1

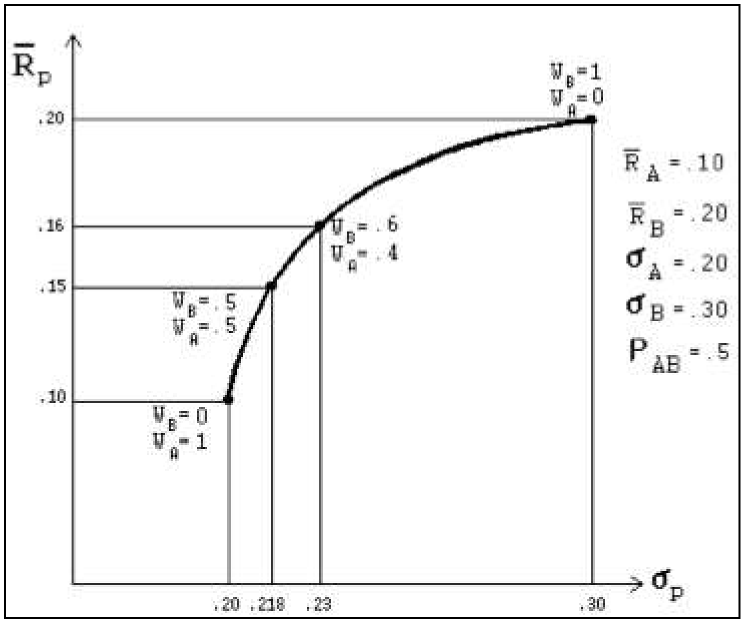


Figure 2: Relationship between portfolio return and risk when ρ_{AB}=.5

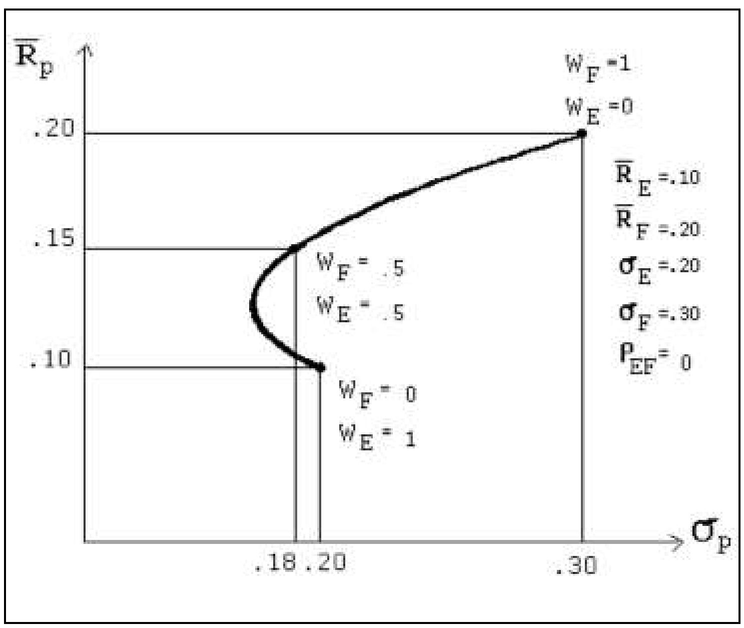


Figure 3: The relationship between portfolio return and risk when ρ_{EF}=0

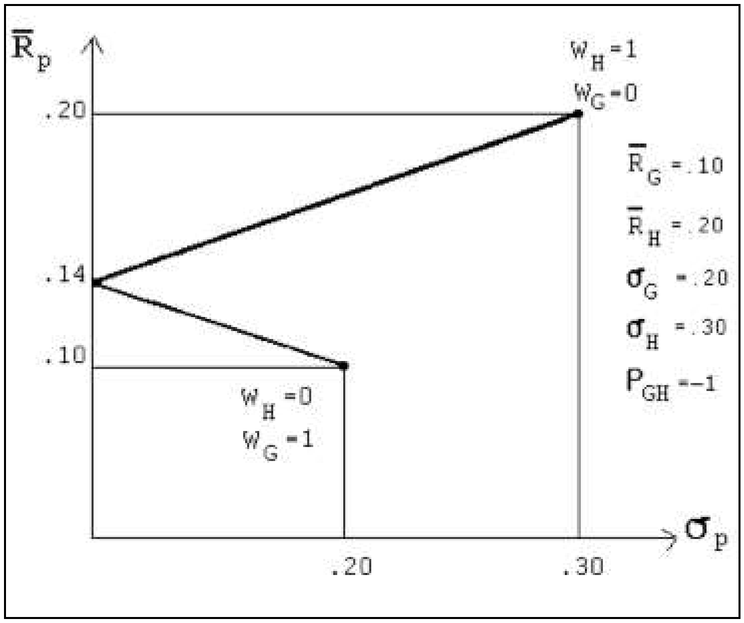


Figure 4: The relationship between portfolio return and risk when ρ_{GH}=-1

Portfolio Diversification: Adding More Securities

In Derivation Box 1, we see how adding additional securities beyond 2 further decreases portfolio risk. Thus, the two keys for diversifying portfolio risk are to select securities with low correlation coefficients with respect to one another and to select many securities. In the next section, we will discuss portfolio efficiency and how it improves as additional securities are added to a portfolio. In fact, one of the most important problems in portfolio management concerns the selection of security weights that minimize portfolio risk at a given return level. For example, suppose that a security analyst has provided a portfolio manager with estimates concerning security expected returns, variances and covariances. The portfolio manager must determine how much to invest in each of the securities, subject to various constraints. If the portfolio manager seeks either to maximize expected portfolio return at a given risk level or minimize portfolio risk at a given expected return level, she will select a portfolio from the Efficient Set, a process that we will discuss later.

From a computational perspective, Equation 4 for portfolio variance becomes unwieldy as we increase the number (n) of securities in the portfolio. This is largely due to the fact that we need to not only consider the returns variance for each of the n securities in the portfolio, we need to consider each of the n^2 returns covariances for each pair of securities i and j in the portfolio. So, for example, a calculation of portfolio variance σ^2 for a 50-security portfolio would need to account for 50 security variances σ_i^2 and 2,500 pairwise covariances $\sigma_{I,j}$ (actually, half that because $\sigma_{i,j}$ i,j = $\sigma_{j,i}$ less the 50 variances). Use of matrix arithmetic is very useful in portfolio math calculations.

Derivation Box 1: Demonstrating That Portfolio Risk Decreases as Portfolio Size Increases

Portfolio diversification results from two conditions: correlations between securities in the portfolio and the number of securities in the portfolio. This proof is concerned with the second condition. First, we define portfolio variance as follows:

(A) $\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_i \sigma_j \rho_{i,j} = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{i,j}$

For sake of simplicity, we assume the following:

1. Our portfolio is equally weighted; that is, $w_i = w_j = 1/n$ for each security weight.
2. All securities have the same variance, σ_i.
3. Each security has the covariance with every other security. This covariance will be equal to the average covariance between pairs of securities.
4. The average security variance in an equally weighted portfolio is larger than the average covariance with other securities. This condition must hold in all cases unless all security returns are perfectly correlated. When all security returns are perfectly correlated, return variances will equal covariances.

These assumptions permit us to rewrite Equation A as follows:

(B) $$\sigma_p^2 = \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} \frac{1}{n} \frac{1}{n} \sigma_{ij} + \sum_{j=1}^{n} \frac{1}{n^2} \sigma_i^2$$

First, note that $\Sigma_{=1}(1/n)\sigma$ is the mean of the security variances. Also, note that there are n terms related to security variances and $n(n-1)$ covariance terms. The average covariance is written:

(C) $$\bar{\sigma}_{i,j} = \sum_{j=1}^{n} \frac{1}{n} \sigma_{i,j}$$

Since the average covariance term will be added $(n-1)$ times, we now write portfolio variance as follows:

(D) $\sigma_p^2 = \sum_{j=1}^{n-1} \frac{1}{n} \bar{\sigma}_{i,j} + \frac{1}{n} \bar{\sigma}_i^2 = \frac{n-1}{n} \bar{\sigma}_{i,j} + \frac{1}{n} \bar{\sigma}_i^2 = \bar{\sigma}_{i,j} - \frac{1}{n} \bar{\sigma}_{i,j} + \frac{1}{n} \bar{\sigma}_i^2 = \bar{\sigma}_{i,j} + n^{-1}\left(\bar{\sigma}_i^2 - \bar{\sigma}_{i,j}\right)$

To demonstrate that portfolio variance decreases as n increases, we merely demonstrate that the derivative of σ_p^2 with respect to n is negative:

(E) $\frac{d\sigma_p^2}{dn} = -n^{-2}\left(\bar{\sigma}_i^2 - \bar{\sigma}_{i,j}\right) < 0$

Derivation Box 1: Demonstrating That Portfolio Risk Decreases as Portfolio Size Increases (cont.)

which will be true whenever the average security variance exceeds the average covariance between non-identical securities. This will be the case whenever the correlation coefficient between security returns is less than one.

Note from Equation D that as the number of securities in the portfolio approaches infinity, the portfolio's risk approaches the average covariance between securities. Thus, only covariance risk is significant for large, well- diversified portfolios. If security returns are entirely independent ($\sigma_{ij} = 0$), portfolio risk approaches zero as the number of securities included in the portfolio tends towards infinity.

Portfolio Arithmetic with Matrices

The portfolio return R_p is simply a weighted average of the individual security returns R_j. We denote the weights by w_i, so that:

$$w_i = (\$\text{invested in } i)/(\$ \text{ invested in } p).$$

If we express the vector of individual returns by $\mathbf{r} = (R_1, \ldots, R_n)^T$ and the vector of weights by $\mathbf{w} = (w_1, \ldots, w_n)^T$, we can express the portfolio return R_p as follows:

$$R_p = \sum_{i=1}^{n} w_i R_i = \mathbf{w}^T\mathbf{r}.$$

Taking the expectation of both sides of this equation, we see that the expected return of the portfolio in terms of the expected value of each of the individual returns satisfies the equation:

$$E[R_p] = \sum_{i=1}^{n} w_i E[R_i].$$

Portfolio return variance is a function of potential portfolio returns and associated probabilities or individual security weights and covariance pairs:

$$\sigma_p^{\,2} = E[(R_p - E[R_p])^2] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \cdot w_j \cdot \sigma_{i,j} = \boldsymbol{w^T V w}$$

where $\mathbf{V} = (\sigma_{i,j})$ is the $n \times n$ matrix of return covariances.

Portfolio Arithmetic with Matrices: Illustration

Suppose that the expected returns of three assets A, B and C are .1, .2 and .3, respectively; $\mathbf{r}^T = [.1, .2, .3]$. Their standard deviations are .2, .3 and .4. The returns covariance between assets A and B is .03, the returns covariance between assets A and C is .04 and the returns covariance between assets B and C is .06. Thus, the variance-covariance matrix for the securities is (recall that the covariance between anything and itself is variance):

$$\boldsymbol{V} = \begin{bmatrix} .04 & .03 & .04 \\ .03 & .09 & .06 \\ .04 & .06 & .16 \end{bmatrix}$$

Let's further suppose that the following weights: $w_A = .625$, $w_B = .25$ and $w_C = .125$ apply to our portfolio; $\mathbf{w}^T = [.625, .25, .125]$. The following equations calculate the expected return and returns variance for our portfolio:

$$\mathrm{E}[r_p] = \boldsymbol{w}^T\boldsymbol{r} = \begin{bmatrix} .625 & .25 & .125 \end{bmatrix} \begin{bmatrix} .1 \\ .2 \\ .3 \end{bmatrix} = .15$$

$$\sigma_p^2 = \boldsymbol{w^T V w} = [.625 \quad .25 \quad .125]\begin{bmatrix} .04 & .03 & .04 \\ .03 & .09 & .06 \\ .04 & .06 & .16 \end{bmatrix}\begin{bmatrix} .625 \\ .25 \\ .125 \end{bmatrix} = .043125$$

Such systems should be much easier to work with, particularly on a spreadsheet than the equations that we expressed earlier.

2. The Efficient Set

This section is concerned with the investor's concern for maximizing expected returns and for minimizing risk. We will first consider, from a theoretical perspective, investor preferences for return and for safety by examining indifference mappings in expected return-risk space. We will then, from both theoretical and practical perspectives, develop the most efficient set of portfolios of risky assets (stocks). Finally, we will supplement the set of risky assets with a riskless asset (bonds) for the purpose of determining the best portfolio for a given investor to invest in.

Indifference Mappings

An indifference curve is a locus of points (curve) representing the various combinations of two choices (perhaps, goods) to which an individual is indifferent. Therefore, an individual receives the same level of satisfaction (or, in economic terms, utility) from all combinations of two goods falling on the same indifference curve. An indifference map is simply a series of one individual's indifference curves.

Indifference curves between two goods that are neither perfect substitutes nor perfect complements generally have the following characteristics:

1. They are downward sloping, indicating that more of either good is preferred to less of that good. Thus, a reduction in the quantity of one good must result in the increase in the quantity of the second good if that individual's utility level is to remain unchanged.
2. They are convex to the origin of the map indicating diminishing marginal utility with respect to both goods. (See Figure 1)

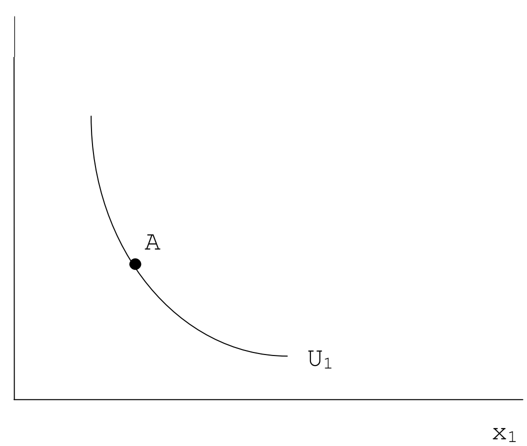


Figure 1: Indifference Curve in Two-Commodity Space

An indifference map, which is composed of an infinite number of indifference curves will have the following attributes:

1. The individual's level of utility will increase as the quantities of both goods increase. Therefore, utility increases as an individual is able to attain combinations on higher indifference curves (curves further to the northwest on the indifference map as in Figure 2).
2. None of the indifference curves on the map will intersect. Intersecting indifference curves violate the property of transitivity. (See Figure 3)

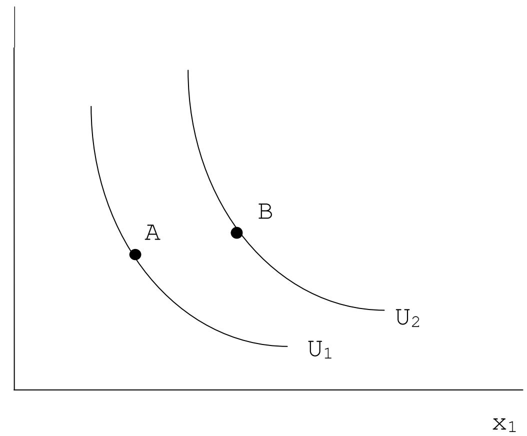


Figure 2: Indifference Map in Two-Commodity Space

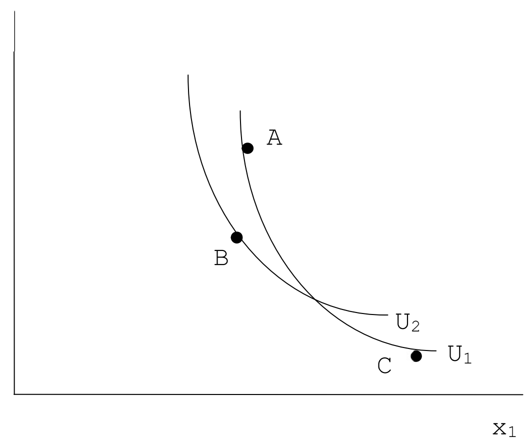


Figure 3: Violation of the Principle of Transitivity[1]

Figure 2 depicts an individual's indifference map for various combinations of two goods, x_1 and x_2. However, consider an indifference map for two commodities, one of which is a good and the other a "bad." For example, the typical investor prefers more return to less return; however, he will prefer less risk to more risk. Thus, return is a good and risk is a "bad". Figure 4 depicts an indifference map for an investor faced with these choices. Notice that since risk is a "bad", the horizontal axis has been reversed. Indifference curves are similarly reversed. The new

[1] Combination B lies on a higher indifference curve than does C, meaning that it is preferred to C. Combination A results in the same utility level as does combination C. Combination A results in more of both goods to the individual than B, thus it must be preferred to B, thus, preferred to C, which cannot be since it is on the same indifference curve. Because of this inconsistency, the principle of transitivity has been violated.

indifference map between a good and a "bad" is similar to a mirror image of the original indifference map between two goods. Now utility increases as safety or the absence of risk increases. Thus, utility increases as standard deviation becomes smaller. Indifference curves are upward sloping, reflecting that increases in risk must result in increases in return in order to maintain constant utility. Furthermore, utility increases as the investor's combination of risk and return levels fall on indifference curves further to the northeast. Thus, the indifference map portrayed by Figure 4 is created by horizontally reversing the map in Figure 2.

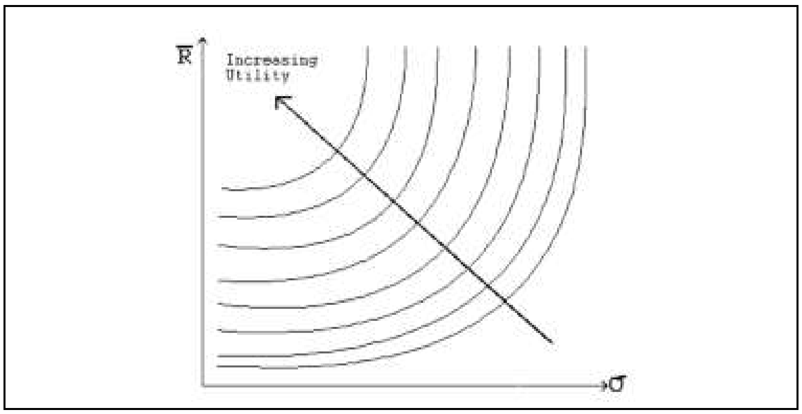
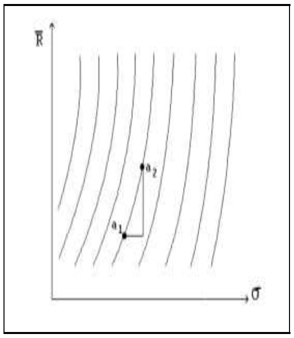


Figure 4: Investor's indifference map for risk and return

An investor's degree of risk aversion is reflected on his indifference map by the slopes of his indifference curves. A highly risk averse investor will have very steep indifference curves, reflecting that, given constant utility, even a small increase in the level of risk he is faced with must result in a substantial increase in his level of return (See Figure 5a). A less risk averse investor will require only a small increase in his return level as compensation for a comparable increase in his risk level. Thus, the slopes of an investor's indifference curves represent his preferences with respect to risk and return. The exact levels of return and risk where an investor chooses to invest will depend on the characteristics of available portfolios as well as his preferences depicted by his indifference map. Thus, in the following sections, we will determine which are the best of all portfolios available to the investor, then use his indifference map to determine which portfolio he will hold. This process is quite similar to the combination of an indifference map and budget constraint in micro-economics.



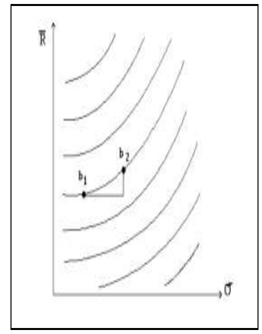


Figure 5.a: Indifference map for extremely risk averse investor.

Figure 5.b: Indifference map for a less risk averse investor.

<u>Developing the Efficient Frontier</u>

We just discussed investor preferences with respect to expected return and risk. Now, we will focus on portfolios made of individual securities. Because investors prefer as much return and as little risk as possible, the most efficient portfolios are those with the following characteristics:

1. Less risk than all portfolios with identical or larger returns and
2. Greater return than all portfolios with identical or less risk.

Efficient portfolios are not dominated by other portfolios. One portfolio dominates a second when one of the following three conditions is met:

1. the first portfolio has both higher return and smaller risk levels than does the second,
2. both portfolios have identical variance but the first portfolio has a higher return level than does the second, or
3. both portfolios have identical returns but the first portfolio has a smaller variance than does the second.

A portfolio is considered dominant if it is not dominated by any other portfolio. Thus, the most efficient portfolios are all dominant.

Consider a market where the average coefficient of correlation between returns on securities is .8. (This is a reasonable assumption in some countries.) For sake of simplicity, assume that there exist in this market five securities, (A) through (E). Combine securities (A) and (B) into a portfolio. Return and risk combinations of the resultant portfolio will fall somewhere on the curve extending between the two securities, depending on their relative weights (See Figure 6). Similarly, securities (B) and (C) can be combined into portfolios as can securities (C) and (D), and (D) and (E) (See Figure 6). We have constructed a series of curves representing risk-return combinations of an infinite number of two security portfolios. These resultant portfolios themselves can be combined into additional portfolios. For example, consider portfolios (AB) and (BC) in Figure 7. These portfolios can be combined into further portfolios as can portfolios (BC) and (CD) as well as (CD) and (DE). The resultant portfolios can all be combined into additional portfolios. Notice that as the portfolios become more diversified, they become more efficient. Thus, the curves representing the risk-return combinations of these portfolios fall further to the northeast on the risk-return space. However, the benefits of this diversification must reach a limit. This is because the portfolios that are being combined are more correlated than the individual securities that they contain. On the curve indicating this limit, further diversification cannot result in more efficient portfolios. The upward sloping portion of this curve is called the Efficient Frontier. The most efficient portfolios of risky assets will have risk-return combinations falling on the efficient frontier. The Efficient Frontier represents the risk-return combinations of the most efficient portfolios in the feasible region. (The feasible region is simply the risk-return combinations of all portfolios available to investors.) Thus, the Efficient Frontier is the left-most, uppermost boundary of the feasible region.

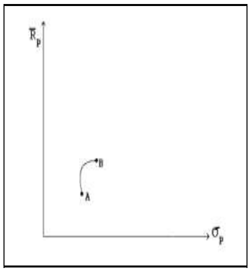


Figure 6: Portfolio risk-return levels when ρ_{AB}=.8

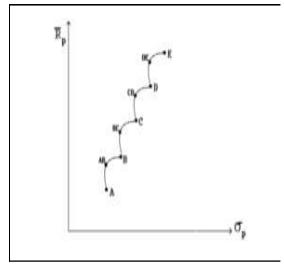


Figure 7: Portfolio risk-return levels when ρ_{ij}=.8 for all *i* and *j*; portfolios are each comprised of two securities A through E.

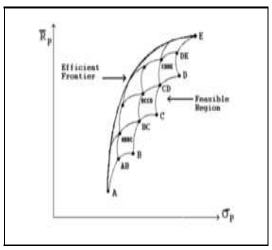


Figure 8: Efficient frontier and feasible region

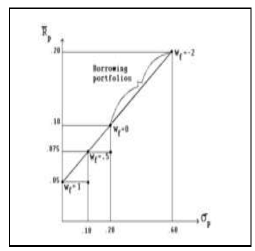


Figure 9: Combinations of risky asset portfolio and the risk-free asset

In reality, there exists no risk-free asset. However, for computational purposes, it is useful to assume the existence of such an asset. Historical evidence suggests that short-term United States Treasury bills have been among the most reliable in actually realizing the returns expected by investors. By purchasing treasury bills, an investor is loaning the government money. The United States government has proven to be an extremely reliable debtor (at least it makes good on all of its treasury bills). Treasury bills are fully backed by the full faith and credit of the U.S. government that has substantial resources due to its ability to tax citizens and create money. Thus, these securities are safer than the safest of corporate bonds or short-term notes. Because the United States Treasury bill seems to be the safest of all investments, its characteristics are often used as surrogates for the characteristics of the risk-free asset.

By definition, the variance (or, standard deviation) of expected returns on the risk-free asset is zero. Thus, an investor purchasing such an asset will certainly receive the return he originally expected. Though this asset is riskless, the investor will require a return, compensating him for inflation and his time value of money. This risk-free rate of return (r_f) can be approximated with the short-term treasury bill rate.

The risk-free asset can be combined with any portfolio of risky assets. Such a portfolio will have a risk-return combination which is simply a weighted average of the risky portfolio's and the risk-free asset's risk-return combinations. For example, consider a portfolio of risky assets with expected return and standard deviation levels of 10% and 20% and a risk-free asset with an expected return of 5%. If the portfolio and the risk-free asset were combined into a new portfolio with equal weights ($w_f = w_m = .5$), the resultant portfolio would have expected return and standard deviation levels of 7.5% and 10%:

$$E\lfloor R_p \rfloor = (.5 \cdot .10) + (.5 \cdot .05) = .075$$

$$\sigma_p = \sqrt{(.5^2 \cdot .20^2) + (.5^2 \cdot 0^2) + 2(.5 \cdot .5 \cdot .1 \cdot 0 \cdot 0)} \qquad \sigma_p = \sqrt{(.25 \cdot .04) + (0) + (0)} = .10$$

Notice that the correlation coefficient between returns on any risky asset and the risk-free asset must be zero. Thus, both portfolio expected returns and portfolio standard deviations will be a linear combination of the individual security returns and standard deviations only when (ρ_{ij}) = 1 or, as in this case, when a risk-free asset is combined with a risky investment.

If an investor has the opportunity to borrow money at the risk-free rate of return (r_f), he has the opportunity to create a negative weight (w_f) for the risk-free asset. For example, if an investor had an initial wealth level of $1000, but wished to invest $3000 in a risky asset with an expected return of 10%, he could borrow $2000 at the risk-free rate of 5% if the lender were certain the investor would fulfill his debt obligation. Since the investor is borrowing money rather than lending (buying treasury bills is, in effect, lending the government money), the weight associated with the risk-free asset is negative. Because the total sum invested in the risky asset is three times as great as the investor's initial wealth level, (w_A) is equal to 3. The investor's expected portfolio return level is 20%, higher than the return of either of the assets comprising the portfolio:

$$E\lfloor R_p \rfloor = (-2 \cdot .05) + (3 \cdot .10) = .20$$

Notice that the sum borrowed is twice as great as the investor's initial wealth level, thus (w_f) is equal to -2. The standard deviation of returns on the portfolio is .6:

$$\sigma_p = \sqrt{(-2^2 \cdot 0^2) + (3^2 \cdot .20^2) + 2(-2 \cdot 3 \cdot 0 \cdot .20 \cdot 0)} \qquad \sigma_p = \sqrt{(3^2 \cdot .20^2)} = \sqrt{.36} = .6$$

Notice that the portfolio standard deviation is higher than the standard deviations of either of the assets comprising the portfolio. Therefore, borrowing money (creating leverage) permits the investor to increase his expected returns; however, he must also face additional risk. Whether an investor will borrow, and exactly how much he will borrow will be discussed later.

Consider an investor who has the opportunity to invest in a combination of a risk-free asset and one of several risky portfolios (A) through (E) depicted in Figure 10. Which of these five portfolios is the best to combine with the risk-free asset? Notice that the portfolios with risk-return combinations on the line connecting the risk-free asset and portfolio (C) dominate all other portfolios available to the investor. Thus, any portfolio whose risk-return combination falls on lines extending through portfolios (A), (B), (D), and (E) will be dominated by some portfolio whose risk-return combination is depicted on the line extending through portfolio(C). This line has a steeper slope than all other lines between the risk-free asset and risky portfolios. The investor's objective is to choose that portfolio of risky assets enabling him to maximize the slope of this line; that is, the investor should pick that portfolio with the largest possible (Θ_p), where

(Θ_p) is defined by Equation (8):

$$\frac{E\lfloor R_p \rfloor - r_f}{\sigma_p} = \Theta_p \quad (8)$$

Therefore, the investor should invest in some combination of portfolio (C) and the risk-free asset. If the curve connecting portfolios (A) through (E) were the Efficient Frontier, then portfolio (C) would be referred to as the market portfolio. This is because every risk averse investor in the market should select this portfolio of risky assets to combine with the riskless asset. Notice that the line extending through portfolio (C) is tangent to the curve at point (C).

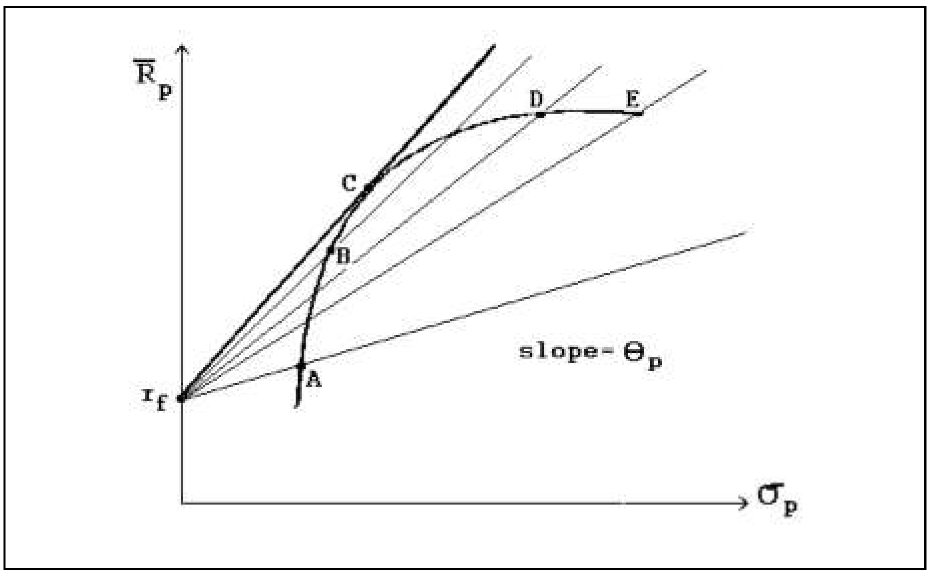


Figure 10: Combination of risk-free asset with one of five portfolios of risky assets

<u>The Capital Market Line</u>

The best portfolio of risky assets to combine with the risk-free security lies on the Efficient Frontier, tangent to the line extending from the risk-free security. This line is referred to as the Capital Market Line (CML). Notice that portfolios on the Capital Market Line dominate all portfolios on the Efficient Frontier. If a risk-free security exists, the Capital Market Line represents risk-return combinations of the best portfolios of securities available to investors. Thus, an investor's risk-return combinations are constrained by the Capital Market Line.

The most efficient portfolio on the Efficient Frontier to combine with the riskless asset is referred to as the Market Portfolio (depicted by [M] in Figure 11). Thus, the Market Portfolio lies at a point of tangency between the Efficient Frontier and the Capital Market Line. All investors should hold portfolios of risky assets whose weights are identical to those of the Market Portfolio. The Capital Market Line combines the Market Portfolio with the riskless asset. This line can be divided into two parts: the lending portion and the borrowing portion. If an investor invests at point (M) on the Capital Market Line, all of his money is invested in the Market Portfolio. If he invests to the left of (M), his portfolio is a lending portfolio. That is, he has purchased treasury bills, in effect, lending the government money, and invested the remainder of his funds in the Market Portfolio. If he invests to the right of point (M), he has a borrowing portfolio. In this case, he has invested all of his funds in the Market Portfolio and borrowed additional money at the risk-free rate to invest in the Market Portfolio. All investors will invest at some risk-return combination on the Capital Market Line. Exactly which risk-return combination an investor will choose will depend on the investor's level of risk aversion.

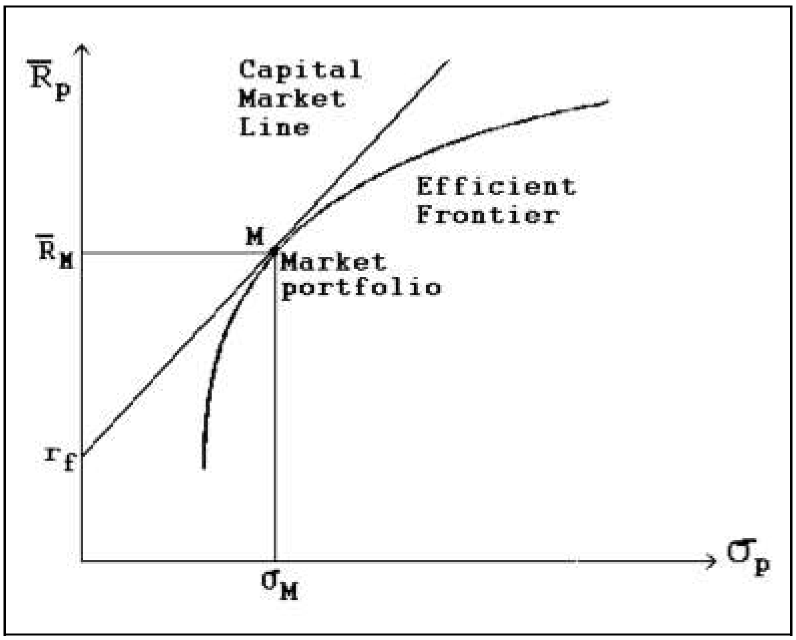


Figure 11: The Capital Market Line

Consider a simple example where there exist two risky securities 1 and 2 in the stock market of Noplacia. A particular investor in this market has projected the following characteristics for these stocks along with an 8% riskless Treasury bill:

$$\sigma_1 = .20$$

$$E[R_1] = .12 \qquad \sigma_2 = .40$$

$$\sigma_{1,2} = -.01$$

$$E[R_2] = .18$$

There also exists a riskless treasury instrument (bill) available for investors of Noplacia. The expected return or implied interest rate on this bill is 8%. Given this interest rate and the above stock projections, determine:

1. the stock weightings for the optimal portfolio of risky securities for this investor
2. the expected return of his portfolio of stocks,
3. the risk of his stock portfolio, as measured by standard deviation,
4. the characteristics of the Capital Market Line faced by this investor; that is, what is the equation for the Capital Market Line?

Since our objective is to select a portfolio of the two risky assets such that the slope of the Capital Market Line is maximized, we will select stock portfolio weights such that θ_p is maximized. To accomplish this, we will find partial derivatives of θ_p with respect to weights of each of the two stocks, set the partial derivatives equal to zero and solve for the weight values w_1 and w_2. We accomplish this in Derivation Box 1.

As we seen in Derivation Box 2, the system of equations we solve to obtain the Capital Market Line in our 2-security economy is given by equation set 9:

$$(9) \qquad \begin{aligned} E[R_1]-r_f &= z_1\sigma_1^2 + z_2\sigma_{1,2} \\ E[R_2]-r_f &= z_1\sigma_{2,1} + z_2\sigma_2^2 \end{aligned}$$

Substituting in appropriate values from our example, we find that:

.

$$\begin{aligned} .12-.08 &= .2^2 z_1 - .01z_2 \\ .18-.08 &= .01z_1 - .4^2 z_2 \end{aligned}$$

that yields:

$$\begin{aligned} .04 &= .04z_1 - .01z_2 \\ .10 &= -01z_1 + .16z_2 \end{aligned}$$

Solving the above simultaneously yields z_1 = 1.174603 and z_2 = .698412. Since $E[R_p]$, r_f and σ_p are the same for both z_1 and z_2, portfolio weights w_1 and w_2 will be linearly related to their z values. Thus, the portfolio weights are determined as follows:

$$\begin{aligned} w_1 &= z_1 \div (z_1 + z_2) = .627 \\ w_2 &= z_2 \div (z_1 + z_2) = .373 \end{aligned}$$

Derivation Box 2: Deriving the Capital Market Line

Since our objective is to select a portfolio of the two risky assets such that the slope of the Capital Market Line is maximized, we will select stock portfolio weights such that θ_p is maximized. To accomplish this, we will find partial derivatives of θ_p with respect to weights of each of the two stocks, set the partial derivatives equal to zero and solve for the weight w_1 and w_2. First, we write Θ_p for the simple two stock portfolio as follows:

(A) $$\Theta_p = \frac{E\left[R_p\right]-r_f}{\sigma_p} = \frac{w_1(E\left[R_1\right]-r_f)+w_2(E\left[R_2\right]-r_f)}{(w_1^2\sigma_1^2+w_2^2\sigma_2^2+2w_1w_2\sigma_{12})^{1/2}}$$

Next, we use the quotient rule to find the derivative of θ_p with respect to w_1 and w_2:

(B) $$\frac{\partial\Theta_p}{\partial w_1} = \frac{\dfrac{\partial(E\left[R_p\right]-r_f)}{\partial w_1}\sigma_p - \dfrac{\partial\sigma_p}{\partial w_1}(E\left[R_p\right]-r_f)}{\sigma_p^2} = 0$$

$$\frac{\partial\Theta_p}{\partial w_2} = \frac{\dfrac{\partial(E\left[R_p\right]-r_f)}{\partial w_2}\sigma_p - \dfrac{\partial\sigma_p}{\partial w_2}(E\left[R_p\right]-r_f)}{\sigma_p^2} = 0$$

However, we need to use the chain rule to find the derivative of the denominator with respect to w_1 and w_2:

(C) $$\frac{\partial\sigma_p}{\partial w_1} = \frac{1}{2}(\sigma_p^2)^{-1/2}(2w_1\sigma_1^2+2w_2\sigma_{12}) = \frac{w_1\sigma_1^2+w_2\sigma_{12}}{\sigma_p}$$

$$\frac{\partial\sigma_p}{\partial w_2} = \frac{1}{2}(\sigma_p^2)^{-1/2}(2w_2\sigma_1^2+2w_1\sigma_{12}) = \frac{w_2\sigma_2^2+w_1\sigma_{12}}{\sigma_p}$$

Notice from equation (A) that the derivative of ($E[R_p]$-r_f) with respect to w_1 equals ($E[R_1]$-r_f). Next, we substitute our results of equation set (C) into equation set (B) making use of the far-right hand side of equation (A):

(D)

$$\frac{\partial\theta_p}{\partial w_1} = \frac{\left(E[R_1]-r_f\right)\sigma_p - \left(E\left[R_p\right]-r_f\right)\left(w_1^2\sigma_1^2+w_2\sigma_{1,2}\right)/\sigma_p}{\sigma_p^2}$$

$$\frac{\partial\theta_p}{\partial w_2} = \frac{\left(E[R_2]-r_f\right)\sigma_p - \left(E\left[R_p\right]-r_f\right)\left(w_2^2\sigma_2^2+w_1\sigma_{2,1}\right)/\sigma_p}{\sigma_p^2}$$

Because the derivatives from Equation Set D are both set equal to zero, we may multiply the numerator by σ_p and maintain the equality. Next, we re-write Equation Set D as follows:

(E) $$E\left[R_1\right]-r_f = \frac{(E\left[R_p\right]-r_f)(w_1\sigma_1^2+w_2\sigma_{12})}{\sigma_p^2}$$

$$E\left[R_2\right]-r_f = \frac{(E\left[R_p\right]-r_f)(w_2\sigma_2^2+w_1\sigma_{12})}{\sigma_p^2}$$

To continue the process of simplification, define the variable z_i as $z_i = w_i\left(E\left[R_p\right]-r_f\right)/\sigma_p^2$ and re-write equation set (D) as follows:

$$E\left[R_1\right]-r_f = z_1\sigma_1^2+z_2\sigma_{1,2}$$

$$E\left[R_2\right]-r_f = z_1\sigma_{1,2}+z_2\sigma_2^2$$

This is the general format for deriving the set of equations needed to solve for characteristics and composition of the Capital Market Line. Extending this system to accommodate more risky securities is straightforward; there will be one z-value and one equation for each risky security.

The return and risk levels of the portfolio (m) of risky stocks are simply:

$$E[R_m] = .627 \cdot .12 + .373 \cdot .18 = .142$$

$$\sigma_m = \left[.627^2 \cdot .04 + .373^2 \cdot .16 + 2 \cdot .627 \cdot .373 \cdot (-.01)\right]^{.5} = .1825$$

Thus the answers to the four problems proposed earlier are as follows:

1. $w_1 = .627$, $w_2 = .373$
2. $E[R_m] = .142$
3. $\sigma_m = .1825$
4. The equation for the Capital Market Line is given as follows:

$$E[R_p] = r_f + \frac{(E[R_m - r_f)]}{\sigma_m} \cdot \sigma_p \quad = .08 + \frac{(.142 - .08)}{.1825}\sigma_p = .08 + .34\sigma_p$$

This process is easily expanded to include as many securities as may exist in the market. Matrix mathematics may simplify computations when the number of securities is large. The Efficient Frontier can be plotted by varying the riskless rate; an additional "market portfolio" is obtained each time a new riskless return is used in the computations.

3. The Capital Asset Pricing Model

The Capital Asset Pricing Model (CAPM) provides us with a theory of equilibrium in capital markets. That is, when the assumptions of the CAPM are met, expected utility maximizing investors will price securities in the market in a manner consistent with the CAPM. The assumptions that underlie the CAPM are:

1. Capital markets are perfect. The assumption of perfect capital markets requires:
    a. zero transactions costs; that is, investors and corporations can buy and sell securities without incurring brokerage fees.
    b. zero personal income taxes.
    c. that investors are able to secure full and costless information.
    d. that no single investor can affect the market price of a security by his transactions.
2. Security returns are normally distributed; therefore, expected utility is a function of expected return and risk levels of an investor's portfolio.
3. All assets are marketable and infinitely divisible.
4. No restrictions are placed on short-sales. (See Glossary)
5. Investors all have identical expectations regarding security expected return and risk levels.
6. There exists a risk-free security with no restrictions on borrowing and lending.
7. Investor planning horizons are one time period.

If these assumptions hold, we can use the Capital Asset Pricing Model to estimate the risk of an investment and also to calculate a discount rate for a present value analysis of that investment. Systematic risk is that portion of a security's risk that is related to variance of the market portfolio. Systematic risk is sensitivity to market portfolio fluctuations. Thus, systematic risk is often referred to as market related risk. Unsystematic risk is that portion of a security's variance that is totally unrelated to risk of the market portfolio; that is, unsystematic risk (or firm-specific or unique risk) is unique to the security under analysis. For example, the potential death of the company's chief executive officer may affect firm performance and may be regarded as a firm-specific or unsystematic risk.

The required return of any security will be related to a risk-free component compensating investors for inflation and their time values of money and premiums compensating investors for both market related and unique risk (though, as we shall see later, unique risk compensation will be zero for a perfectly diversified portfolio). Market related risk of a security can be measured as the standard deviation of returns associated with that security relative to the standard deviation of returns associated with the market portfolio multiplied by the coefficient of correlation between returns on the security and the market portfolio:

(8) $$\beta_i = \frac{\sigma_i}{\sigma_m}\rho_{i,m}$$

Thus, Beta (β_i) measures the risk of security (i) relative to the risk of the market portfolio. The coefficient of correlation relates to the reduction in portfolio risk realized by including security (i) in a well- diversified portfolio. If the standard deviation of expected returns on security (A) and the market portfolio were .4 and .2, respectively, and the correlation coefficient between returns on the two securities were .75, the Beta (β_A) of security (A) would be 1.5:

$$\beta_A = \frac{\sigma_A}{\sigma_M}\cdot\rho_{A,M} = \frac{.4}{.2}\cdot 0.75 = 1.5$$

Notice that Equation (8) can be rewritten as follows:

(9) $$\beta_i = \frac{\sigma_i\sigma_m\rho_{i,m}}{\sigma_m^2} = \frac{COV(i,m)}{\sigma_m^2}$$

Betas are normally computed on the basis of historical standard deviations and correlation coefficients. Many investors believe that the relative stability over time of these statistical measures justifies the use of "historical" betas. Historical betas are computed on the basis of historical standard deviations and correlation coefficients. Frequently, analysts will compute historical stock betas, standard deviations and correlation coefficients based on five years of historical monthly returns data. Several investment advisory services such as Value Line will provide historical betas for a large number of widely traded stocks.

By definition, the market portfolio (or "average" security) requires a risk premium of (r_m - r_f). Therefore, an investor requires this premium to compensate for the risk associated with the "average" security. In fact, the systematic risk premium required by any investor for any security (i) is:

(10) $$(rr_i - r_f) = \beta_i(r_m - r_f)$$.

Notice that, since ($\sigma_{m,m}$) = (σ_m^2), and ($\rho_{m,m}$) = 1, the beta of the market portfolio equals one.

If an investor assumes additional unsystematic (unique) risk by the purchase of a security, he will require an unsystematic risk premium. This premium will be unrelated to the market portfolio; thus, it will be unique to each individual security. The total risk premium required by an investor for the purchase of security (A) will be:

(11) $$(rr_A - r_f) = \beta_A(r_m - r_f) + \mu_A \quad .$$

Covariances between the non-market related return components between securities in a well-diversified portfolio will, on average, equal zero. We find, using time-series data, that on average, securities earn returns equal to the risk-free return plus their required systematic risk premium components. Therefore, the unsystematic risk premium required by an investor holding a well-diversified portfolio will be zero. We can ignore the (μ_A) component of Equation (11) for shareholders with well-diversified portfolios. Therefore, the return required by any investor to purchase any security (i) will be:

(12) $$rr_i = r_f + \beta_i(r_m - r_f)$$

This is the Capital Asset Pricing Model. The CAPM enables us to determine the required return for any investment, given its risk characteristics and the current risk-free rate of return. Thus, an investor must expect to receive the required return on an investment in order to purchase it. If capital markets function according to the assumptions outlined earlier, security prices will be such that the expected returns on all securities will equal their required returns. Although many of these assumptions do not hold in reality, the CAPM still provides us with good approximations of required security returns. Nonetheless, there exist a number of variations of the CAPM that have been adjusted to adapt to more realistic market conditions.

If the required return on the market portfolio were .14 and the current risk-free rate were .06, the required return for security (A) described above would be .18:

$$rr_A = .06 + 1.5(.14 - .06) = .18 \quad .$$

The required rate of return generated by the Capital Asset Pricing Model provides an excellent risk-adjusted discount rate useful for evaluating a variety of investments. This risk- adjusted discount rate reflects inflation and investors' time value of money through its risk-free component. The CAPM reflects investors' risk-return preferences through the market systematic risk premium ($r_m - r_f$). Furthermore, the model reflects the risk of the security under evaluation through the Beta (β_i) component. Therefore, any security can be evaluated by determining its Beta (β_i), plugging it into the CAPM, and then discounting projected cash flows using the required rate of return as a discount rate:

(8) $$\beta_i = \frac{\sigma_i}{\sigma_m}\rho_{i,m}$$

(12) $$rr_i = r_f + \beta_i(r_m - r_f)$$

(13) $$PV_i = \sum_{t=1}^{n} \frac{CF_t}{(1 + rr_i)^t}$$

Thus, if security (A) were expected to pay a \$10 dividend in each of the next three years and then be sold for \$100, its current value would be \$82.61:

$$PV_A = \frac{\$10}{(1+.18)^1} + \frac{\$10}{(1+.18)^2} + \frac{\$10 + \$100}{(1+.18)^3} = \$82.61 \quad .$$

The Securities Market Line

The Securities Market Line (Figure 12) is a graphical representation of the Capital Asset Pricing Model. The required returns for securities are plotted against their risk levels (β_i). If the market is pricing securities efficiently, every security will have an expected return corresponding to its Beta level on the Securities Market Line. If the market does exhibit pricing inefficiencies, expected security returns may differ from their required levels. Securities whose expected returns exceed their required levels will earn higher than normal profits and therefore should be purchased. Securities whose expected returns fall below the Securities Market Line will earn less than normal profits and thus should be sold. If, in an efficient market, investors are buying those securities whose returns lie above the SML, this buying pressure will cause the purchase prices of these securities to rise, thereby decreasing their returns. This buying pressure will force returns back to the Securities Market Line. Similarly, securities whose returns lie beneath the SML will produce selling action, bidding down prices and forcing returns back up to the Securities Market Line.

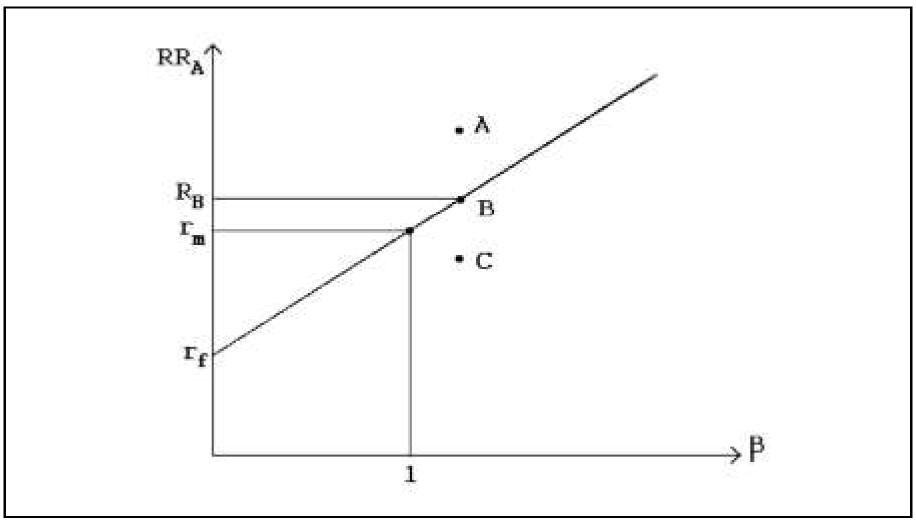


Figure 12: Securities Market Line.

Note: Security A is underpriced, generating too high a return. Security C is overpriced. Thus, every investor should want to purchase A, bidding up its price, sell C, bidding down its price until the returns of both are equal to R_B.

The beta of a portfolio is simply a weighted average of the betas of its component securities. According to the Capital Asset Pricing Model Theory, one can replicate the systematic risk of any investment by constructing a portfolio with security weights such that the portfolio beta equals the investment beta. If the expected return on the portfolio is not exactly the same as the expected return on the investment, one can obtain a "free" return. For example, if the return on the investment exceeds that of the portfolio with the same beta, one should sell the portfolio with the low return to raise cash to purchase the investment with the higher return. No net cash flow is used because the sale proceeds were used to finance the purchase. No risk has been assumed because the beta on the sold portfolio was exactly the same as the beta of the purchased investment. Yet since the sold portfolio return was lower than that of the purchased investment, a profit has been obtained at no cost. Such a transaction is called a profitable arbitrage transaction - it involves a profit with zero net investment and zero risk. In this case, unsystematic risk is assumed to be unimportant because it can be diversified away.

Deriving the Capital Asset Pricing Model

Earlier, we discussed the derivation of the Efficient Frontier and the Capital Market Line based on the assumed desire of investors to minimize portfolio variance while maximizing returns. Thus, our problem was to maximize θ_p, the slope of a line with vertical intercept equal to r_f and tangent to the Efficient Frontier:

(1) $$MAX\ \Theta_p = \frac{E\lfloor R_p \rfloor - r_f}{\sigma_p} \quad by \quad \frac{\partial \Theta_p}{\partial w_i} = 0 \quad \forall i$$

Now, define the term $\lambda = (E[R_m - r_f)/\sigma_p^2$ as the market price of risk in that it represents the risk premium required by investors for a given level of variance of returns on portfolio p. The series of portfolio weights w_i which maximize Θ_p represent weights of the market portfolio. As we saw earlier in our derivation of the Capital Market Line, we determine this series of portfolio weights through the use of the Chain and Product Rules, obtaining a system of equations with the following format for each of n securities i:

(2) $$E[R_i] - r_f = \lambda[w_1\sigma_{1,i} + w_2\sigma_{2,i} + \cdots + w_i\sigma_i^2 + \cdots + w_n\sigma_{n,i}]$$

As an aside, we define the market return and the covariance between returns on the market and Security i as follows:

(3) $$E[R_m] = w_1E[R_1] + w_2E[R_2] + \cdots + w_nE[R_n]$$

(4) $$\sigma_{i,m} = E[(R_i - E[R_i]) \times (w_1(R_1 - E[R_1]) + \cdots + w_n(R_2 - E[R_2]))]$$

This covariance between returns on Security i and on the market is written in expectations format as follows. Moving the expectations operators inside of the brackets yields:

(5) $$\sigma_{i,m} = [w_1E(R_i - E[R_i])(R_1 - E[R_1])] + \cdots + [w_nE(R_i - E[R_i])(R_n - E[R_n])]$$

Note that the inside terms are covariances, enabling us to rewrite Equation (6) as follows:

(6) $$\sigma_{i,m} = [w_1\sigma_{1,i} + w_2\sigma_{2,i} + \cdots + w_i\sigma_i^2 + \cdots + w_n\sigma_{n,i}]$$

With Equation (6) defining the covariance between returns on Security i and the market portfolio, we can rewrite Equation (2) as follows:

(7) $$E[R_i] - r_f = \lambda\sigma_{i,m}$$

which is easily rewritten as follows:

$$(8)\quad E[R_i] = r_f + \frac{E[R_m]-r_f}{\sigma_m^2}\sigma_{i,m} = r_f + \frac{\sigma_{i,m}}{\sigma_m^2}\left(E[R_m]-r_f\right) = r_f + \beta_i\left(E[R_m]-r_f\right)$$

CAPM: Second Derivation

The CAPM is concerned with the pricing relationships among securities and the relationships among expected security returns r. The investor portfolio problem is to select n portfolio weights w_i so as to minimize portfolio risk σ_p^2subject to some target return r_p as follows:[1]

$$\min_{\boldsymbol{w}} \sigma_p^2 = \boldsymbol{w^T V w}$$
$$\boldsymbol{w}^T\boldsymbol{r} = r_p$$
$$\boldsymbol{w}^T\boldsymbol{\iota} = 1$$

This problem is solved with the following Lagrangian:

$$L = \boldsymbol{w^T V w} - \boldsymbol{\lambda_1}\left(\boldsymbol{w}^T\boldsymbol{r} - r_p\right) - \boldsymbol{\lambda_2}(\boldsymbol{w}^T\boldsymbol{\iota} - 1)$$

with first-order conditions as follows:

$$\boldsymbol{\nabla}\mathrm{L}(\mathbf{w}) = 2\mathbf{Vw} - \lambda_1\mathbf{r} - \lambda_2\boldsymbol{\iota} = \mathbf{0}$$

$$\frac{\partial \mathrm{L}}{\partial \lambda_1} = \mathbf{w}^{\mathrm{T}}\mathbf{r} - \mathrm{r_p} = 0$$

$$\frac{\partial \mathrm{L}}{\partial \lambda_2} = \mathbf{w}^{\mathrm{T}}\boldsymbol{\iota} - 1 = 0$$

The first-order conditions implied by the gradient can be expressed based on the second line given the following LaGrangian:

$$L = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j} - \boldsymbol{\lambda_1}\left(\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{w_i}\,\boldsymbol{E}[\boldsymbol{R_i}] - r_p\right) - \boldsymbol{\lambda_2}\left(\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{w_i} - 1\right)$$

$$\frac{\partial L}{\partial w_i} = 2\sigma_{i,1}w_1 + 2\sigma_{i,2}w_2 + \ldots + 2\sigma_{i,n}w_n - E[r_i]\lambda_1 - \lambda_2 = 0$$

And then simplified as follows:

[1] $\boldsymbol{V}$ is the security variance-covariance matrix, $\boldsymbol{w}$ is a vector of portfolio weights, $\boldsymbol{w^T}$ is the transpose of that vector and $\boldsymbol{\iota}$ is the unit vector.

$$2\sum_{j=1}^{n}\sigma_{i,j}w_j = E[r_i]\lambda_1 - \lambda_2$$

For, example, since the riskless asset f has zero variance and zero covariance with respect to all other assets, its first order condition can be written as follows:

$$0 = E[r_f]\lambda_1 - \lambda_2$$

Subtracting these two first-order conditions leads to the following:

$$2\sum_{j=1}^{n}\sigma_{i,j}w_j = E[r_i - r_f]\lambda_1$$

Because of the following equality:[2]

$$\sum_{j=1}^{n}\sigma_{i,j}w_j = \sigma_{i,m}$$

We can write the following for security i and market portfolio m:

$$2\sigma_{m,i} = E[r_i - r_f]\lambda_1$$

$$2\sigma_{m,m} = E[r_m - r_f]\lambda_1 = 2\sigma_m^2$$

This means that we can solve for λ_1 as follows:

$$\lambda_1 = \frac{2\sigma_m^2}{E[r_m - r_f]}$$

Now, substitute this into earlier equations:

$$2\sum_{j=1}^{n}\sigma_{i,j}w_j = 2\sigma_{m,i} = E[r_i - r_f]\lambda_1 = E[r_i - r_f]\frac{2\sigma_m^2}{E[r_m - r_f]}$$

$$2\sigma_{m,i} = E[r_i - r_f]\frac{2\sigma_m^2}{E[r_m - r_f]}$$

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[2] This equality is obtained as follows: $\sigma_{i,m} = E[(R_i - E[R_i]) \times (w_1(R_1 - E[R_1]) + \cdots + w_n(R_2 - E[R_2]))]$. Moving the expectations operators inside of the brackets yields: $\sigma_{i,m} = [w_1 E(R_i - E[R_i])(R_1 - E[R_1])] + \cdots + [w_n E(R_i - E[R_i])(R_n - E[R_n])]$. The inside terms are covariances, enabling us to rewrite this equation as follows: $\sigma_{i,m} = [w_1\sigma_{1,i} + w_2\sigma_{2,i} + \cdots + w_i\sigma_i^2 + \cdots + w_n\sigma_{n,i}]$. Also, w was substituted for x.

$$E[r_i - r_f] = \frac{\sigma_{m,i}}{\sigma_m^2} E[r_m - r_f]$$

Where $\beta_i = \frac{\sigma_{m,i}}{\sigma_m^2}$, this final expression represents the Capital Asset Pricing Model. The CAPM enables us to determine the required return for any investment, given its risk characteristics and the current riskless rate of return. Thus, an investor must expect to receive the required return on an investment in order to purchase it. If capital markets function according to the assumptions outlined earlier, security prices will be such that the expected returns on all securities will equal their required returns. Although many of these assumptions do not hold in reality, the CAPM still provides us with good approximations of required security returns. Nonetheless, there exist a number of variations of the CAPM that have been adjusted to adapt to more realistic market conditions.

Portfolio Selection Illustration

Suppose that the expected returns of the three assets A, B and C in an economy are .1, .2 and .3, respectively. Their standard deviations are .2, .3 and .4. The returns covariance between assets A and B is .03, the returns covariance between assets A and C is .04 and the returns covariance between assets B and C is .06. Assuming that the Capital Asset Pricing Model holds in this economy, we want to derive some financial market characteristics. To begin with, we want to determine the weights of this optimal or lowest risk portfolio if based on a target portfolio return of 15%, which equals the MRT (Marginal Rate of Transformation) for this economy.

First, since the expected portfolio return is .15 and the weights must sum to 1, the LaGrange function is written as either of the following:

$$\begin{aligned} L = {} & .04w_A^2 + .09w_B^2 + .16w_C^2 + 2\cdot.03w_Aw_B + 2\cdot.04w_Aw_C + 2\cdot.06w_Bw_C \\ & + \lambda_1(.15 - .1w_A - .2w_B - .3w_C) + \lambda_2(1 - w_A - w_B - w_C) \end{aligned}$$

or:

$$\begin{aligned} \mathrm{L} = {} & [w_A \quad w_B \quad w_C]\begin{bmatrix} .04 & .03 & .04 \\ .03 & .09 & .06 \\ .04 & .06 & .16 \end{bmatrix}\begin{bmatrix} w_A \\ w_B \\ w_C \end{bmatrix} \\ & -\lambda_1[w_A \quad w_B \quad w_C]\begin{bmatrix} .1 \\ .2 \\ .3 \end{bmatrix} + .15\lambda_1 - \lambda_2[w_A \quad w_B \quad w_C]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1\lambda_2 \end{aligned}$$

First order conditions are follows, which combines $\boldsymbol{\nabla}\mathrm{L}(\mathbf{w}) = \mathbf{0}$, $\frac{\partial \mathrm{L}}{\partial \lambda_1} = \mathbf{w}^{\mathrm{T}}\mathbf{r} - r_p = 0$ and $\frac{\partial \mathrm{L}}{\partial \lambda_2} = \mathbf{w}^{\mathrm{T}}\boldsymbol{\iota} - 1 = 0$, the latter two of which are slightly rearranged:

$$\underset{\mathbf{V^*}}{\begin{bmatrix} .08 & .06 & .08 & -.1 & -1 \\ .06 & .18 & .12 & -.2 & -1 \\ .08 & .12 & .32 & -.3 & -1 \\ -.1 & -.2 & -.3 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 \end{bmatrix}} \underset{\mathbf{w^*}}{\begin{bmatrix} w_A \\ w_B \\ w_C \\ \lambda_1 \\ \lambda_2 \end{bmatrix}} = \underset{\mathbf{o^*}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ -.15 \\ -1 \end{bmatrix}}$$

We invert Matrix **V*** then solve for Vector **w***. We find the following weights: w_A = .625, w_B = .25 and w_C = .125; our LaGrange multipliers are λ_1 = .225 and λ_2 = .0525. The expected return and standard deviation of portfolio returns are .15 and .207665, respectively:

$$E[r_p] = \boldsymbol{w}^T\boldsymbol{r} = [.625 \quad .25 \quad .125]\begin{bmatrix}.1\\.2\\.3\end{bmatrix} = .15$$

$$\sigma_p^2 = \boldsymbol{w^T V w} = [.625 \quad .25 \quad .125]\begin{bmatrix}.04 & .03 & .04\\.03 & .09 & .06\\.04 & .06 & .16\end{bmatrix}\begin{bmatrix}.625\\.25\\.125\end{bmatrix} = .043125$$

Based on a step from our CAPM derivation, we can solve the following for this economy's riskless return rate r_f:

$$E[r_m - r_f]\lambda_1 = 2\sigma_m^2$$

$$r_f = -\frac{2\sigma_m^2}{\lambda_1} + E[r_m] = -\frac{2\times.043125}{.225} + .15 = -.2333$$

Note that this is an odd result with a negative interest rate, but, still consistent with the Capital Asset Pricing Model.

Suppose that we wanted to calculate the covariance between returns on security A and returns on the market. Again, based on a step in the CAPM derivation, we do so as follows:

$$2\sigma_{m,i} = E[r_i - r_f]\lambda_1$$

$$\sigma_{m,A} = \frac{E[r_A - r_f]\lambda_1}{2} = \frac{E[.1 + .23333] \times .225}{2} = .037496$$

Since $\beta_i = \frac{\sigma_{m,i}}{\sigma_m^2}$, the beta of security A is .037496/.043125 = .869478 = β_A.

CAPM: Empirical Tests

Does the CAPM work? More precisely, to what extent does the Capital Asset Pricing Model explain the variation structure of security returns? Most of the earlier tests of the CAPM were quite supportive of the model, finding linear relationships between stock and market returns, statistically significant betas and returns being unrelated to unsystematic returns (e.g., Sharpe and Cooper [1972], Black, Jensen and Scholes [1973], Miller and Scholes [1972]). However, Roll [1977] made a very convincing argument of the untestability of CAPM, citing, among other issues, our inability to correctly discern the market portfolio and its composition, being a critical element of every test. Fama and French [1992] were more critical of CAPM, find that the relationship between security returns and beta is at best very weak after 1963. Using firm size and Book to Market Equity values, Fama and French found that the significance of beta disappears in cross-sectional tests with returns. Nevertheless, the empirical strength of CAPM remains a controversial issue among both academics and practitioners alike.

4. Mathematics Underlying Arbitrage Pricing Theory

Arbitrage Pricing Theory is a theory of capital markets equilibrium; that is, it is a theory that describes the relationships among security prices under certain assumptions, among which is the condition that security returns are linearly related to a set of independent factors (which might be related to the market, economy, industries or other source of covariance among security returns). Deriving the APT will require some mathematics preparation.

Underlying Mathematics: Orthogonal Vectors

Two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be orthogonal when $\mathbf{x}^T\mathbf{y} = 0$. A set of orthogonal vectors is linearly independent. Geometrically, the vectors $\mathbf{x}$ and $\mathbf{y}$ will represent perpendicular lines (lines placed at right angles with respect to one another). Symbolically, we write $\mathbf{x} \perp \mathbf{y}$ to mean that vector $\mathbf{x}$ is orthogonal to vector $\mathbf{y}$.

One result from linear mathematics that is extremely useful in finance follows from two assumptions:

1. Assume that each vector $\mathbf{x}$ in a set {X} of *n-1* vectors $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{n-1}\}$ is orthogonal to some other vector $\mathbf{w}$. That is, $\mathbf{x}_1{}^T\mathbf{w} = 0$, $\mathbf{x}_2{}^T\mathbf{w} = 0$ and so on.
2. Furthermore, assume that *because* each of the *n-1* vectors in X is orthogonal to $\mathbf{w}$, an n^{th} Vector $\mathbf{x}_n$ must also be orthogonal to Vector $\mathbf{w}$. Thus, $\mathbf{x}^T\mathbf{w} = 0$. That is, assume that the subset of *n-1* vectors from set {X} being orthogonal to $\mathbf{w}$ implies that Vector $\mathbf{x}_n$ must also be orthogonal to Vector $\mathbf{w}$.

Result: Vector $\mathbf{x}_n$ must be a linear combination of vectors $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{n-1}\}$.

Thus, in order for the orthogonality between each vector in set X and Vector $\mathbf{w}$ to imply orthogonality between Vector $\mathbf{x}_n$ and Vector $\mathbf{w}$, Vector $\mathbf{x}_n$ must be a linear combination of vectors $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{n-1}\}$. In order for a set of n-1 vectors to make this statement regarding Vector $\mathbf{x}_n$, $\mathbf{x}_n$ must be a linear combination of the n-1 vectors that imply that relationship between $\mathbf{x}_n$ and $\mathbf{w}$.

Consider the following example where vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{w}$ are given as follows:

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad c = \begin{bmatrix} 0 \\ 2 \\ 5 \\ 0 \\ 1 \end{bmatrix} \qquad d = \begin{bmatrix} .07 \\ .13 \\ .27 \\ .05 \\ .09 \end{bmatrix} \qquad w = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

Vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are all orthogonal to Vector $\mathbf{w}$. That is, pre-multiplication of any one of vectors $\mathbf{a}$, $\mathbf{b}$ or $\mathbf{c}$ vector $\mathbf{w}^T$ will result in zero: $\mathbf{w}^T\mathbf{a} = 0$, $\mathbf{w}^T\mathbf{b} = 0$, $\mathbf{w}^T\mathbf{c} = 0$. Suppose that the orthogonality of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ to $\mathbf{w}$ imply that $\mathbf{w}^T\mathbf{d} = 0$. Then vector $\mathbf{d}$ must be a linear combination of vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Thus, $\mathbf{a}\perp\mathbf{w}$, $\mathbf{b}\perp\mathbf{w}$, $\mathbf{c}\perp\mathbf{w}$ and $\mathbf{d}\perp\mathbf{w}$, and we can write $\mathbf{d}$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, or more specifically as follows:

$$\mathbf{d} = (.05 \times \mathbf{a}) + (.02 \times \mathbf{b}) + (.04 \times \mathbf{c})$$

Thus, Vector **d** is a linear combination of vectors **a**, **b**, and **c** because these three vectors imply that $\mathbf{d} \perp \mathbf{w}$. Similarly, if Vector **d** must be orthogonal to Vector **w** whenever vectors **a**, **b**, and **c** are orthogonal to Vector **w**, Vector **d** must be a linear combination of vectors **a**, **b**, and **c**. The last statement provides that mathematical foundation to the Arbitrage Pricing Theory.

5. Deriving the APT Model

Underlying Assumptions

1. Capital markets are competitive and frictionless.
2. Investors share homogeneous expectations.
3. Security returns are generated by the following factor model, where **r** is an $n \times 1$ vector of security returns in an n security economy, $\boldsymbol{\beta}$ is an $n \times (k+1)$ matrix of factor loadings in an economy where security prices are linearly related to k factors or indices, **f** is a $(k+1) \times 1$ vector of random factor values (a factor value is the difference between the actual index value and its expected value) prevailing during the time during which returns are measured, **e** is an $n \times 1$ vector of residual return components for securities and f_j is the j'th index that generates security returns:

$$E\left[\widetilde{R}_i\right] = b_{i0}\widetilde{f}_0 + b_{i1}\widetilde{f}_1 + b_{i2}\widetilde{f}_2 + \ldots + b_{ik}\widetilde{f}_k + \widetilde{\varepsilon}_i$$

or

$$\mathbf{r} = \boldsymbol{\beta}\mathbf{f}$$

where:

$$\widetilde{f} = \left[I_j - E\left(I_j\right)\right]$$

$$E\left[\widetilde{f}_j\right] = 0$$

$$E\left[\widetilde{f}_j \cdot \widetilde{f}_h\right] = 0 \text{ for } j \neq h$$

$\widetilde{R}_i$ is a random return for security i,

$$E\left[\widetilde{\varepsilon}_i\right] = 0 = E\left[\widetilde{\varepsilon}_i \cdot \widetilde{\varepsilon}_g\right]$$

4. The number of securities, n, exceeds the number of factors, *k*.
5. Enough securities exist to diversify away unsystematic risk. Thus, $\mathbf{w}^T\mathbf{e} = 0$.
6. Arbitrage opportunities cannot persist. Therefore, investors will not be able to form zero net investment, zero-risk portfolios which earn a positive profit. The zero-net investment, zero risk portfolio will have the following characteristics:

a. $\sum_{i=1}^{n} w_i = 0$ or $\mathbf{w}^T\boldsymbol{\iota} = 0$: Zero net Investment

$$\sum_{i=1}^{n} w_i \cdot b_{i1} = 0$$

b. $\vdots$ or $\mathbf{w}^T\boldsymbol{\beta} = [\mathbf{0}]$: Zero Factor or systematic risk

$$\sum_{i=1}^{n} w_i \cdot b_{ik} = 0$$

c. $$\sum_{i=1}^{n} w_i \cdot \tilde{\varepsilon}_i = 0$$ or $\mathbf{w}^T\boldsymbol{\varepsilon} = 0$: zero unique risk

The sixth assumption implies that the following zero return condition must hold for any zero-net investment, zero-risk arbitrage portfolio:

$$\tilde{R}_p = \sum_{i=1}^{n} w_i E\left[\tilde{R}_i\right] + \sum_{i=1}^{n} w_i b_{i1} \tilde{f}_1 + \ldots + \sum_{i=1}^{n} w_i b_{ik} \tilde{f}_k + \sum_{i=1}^{n} w_i \varepsilon = 0$$

or

$$\tilde{R}_p = \boldsymbol{w}^T \boldsymbol{r} + \boldsymbol{w}^T \boldsymbol{\beta} \tilde{\boldsymbol{f}} + \boldsymbol{w}^T \boldsymbol{\varepsilon} = 0$$

where $\mathbf{r}$ is the $n \times 1$ security expected returns vector. Since there is no risk on this arbitrage portfolio, $\boldsymbol{w}^T \boldsymbol{\beta} \tilde{\boldsymbol{f}} + \boldsymbol{w}^T \boldsymbol{\varepsilon} = 0$, and the security expected returns vector is orthogonal to the weights vector:

$$E[R_p] = \Sigma_{i=1} w_i E[R_i] = \mathbf{w}^T\mathbf{r} = 0$$

The Equation a from Assumption 6 above states that the weights vector is orthogonal to the unit vector. The Equations b from Assumption 6 state that the weights vector is orthogonal to each of the beta coefficients vectors. Equations c from Assumption 6 state that the weights vector is orthogonal to the unique return components vector. Thus, these three sets of equations define the arbitrage portfolio — the zero net investment risk-free portfolio. The no-arbitrage assumption states that the return on this arbitrage portfolio must equal zero. Because orthogonality in the first three sets of vectors implies orthogonality in the fourth set of vectors, the returns vector must be a linear combination of the unit, betas and epsilons vectors.

<u>The APT</u>

Because orthogonality in the first three sets of vectors implies orthogonality in the fourth set of vectors, the returns vector must be a linear combination of the unit, betas and epsilons vectors. Thus, the securities expected returns vector is orthogonal to the weights vector. Therefore, the securities expected return vector must be a linear combination of the unit, beta coefficients vector and unique risk components vector (though, the unique risk components has an expected value of zero). Thus:

$$E[R_i] = \delta_0 + \delta_1 b_{i1} + \ldots + \delta_k b_{ik}$$ <u>The APT Model</u>

$$\mathbf{E[r]} = \delta^T\beta$$

where the δ values may be regarded as risk premia or coefficients relating the betas to expected returns. Thus, these δ values may be interpreted as follows:

$\delta_0 = r_f$ the riskless return rate
$\delta_i = f_i - r_f$ factor risk premia for i = 1 to k.

Factor risk premia δ_i may be described as the return on a portfolio with $b_j = 0$ for all indices except index i where b_i=1.

<u>APT Illustration</u>

Consider the following example based on a two-index model that generates security returns in a particular economy:

$$E[R_i] = a_i + b_{i,1}f_1 + b_{i,2}f_2$$

Suppose that a given portfolio A has an expected return of .13, a beta with index 1 equal to 2 and has zero covariance to the second index. Portfolio B has an expected return of .11, is uncorrelated with the first index and has a beta of 6 with the second index. Portfolio C has an expected return of .10 and has betas equal to one with each of the two indices.

We can write the expected returns Vector **r**, the beta Matrix **β**, the unit Vector **i** and the factor risk premia Vector **δ** as follows:

$$E[\vec{r}\,] = \begin{bmatrix} .13 \\ .11 \\ .10 \end{bmatrix} \quad \beta = \begin{bmatrix} 2 & 0 \\ 0 & 6 \\ 1 & 1 \end{bmatrix} \quad \vec{i} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

The expected returns vector might be expressed as follows:

$$E[\vec{r}\,] = \begin{bmatrix} .13 \\ .11 \\ .10 \end{bmatrix} = \delta_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 6 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

We can solve the above for factor risk premia δ_0, δ_1 and δ_2. Alternatively, this system might be expressed as a system of equations as follows:

$$0.13 = 1\delta_0 + 2\delta_1 + 0\delta_2$$
$$0.11 = 1\delta_0 + 0\delta_1 + 6\delta_2$$
$$0.10 = 1\delta_0 + 1\delta_1 + 1\delta_2$$

The riskless return rate and two-factor risk premia are determined to be as follows: $\delta_0 = .05 = r_f$, $\delta_1 = .04$, $\delta_2 = .01$.

Based on the above information, what would be the value of a security whose expected total payoff in one period is \$100 assuming that its Beta (b_1) with δ_1 is 2 and its Beta (b_2) with δ_2 is .8? First, we find its expected return to be (.05 + 2 × .04 + .8 × .01) = .20. Thus, its current value is \$100/1.2 = 83.333.

6. Index Models and Applications of APT

The Multi-Index Model enables the analyst to attribute multiple sources of covariance between security returns. For example, the analyst might believe that the return structures among securities can be attributed to m significant sources I_j. These m significant sources of commonality among security returns, I_j could be any set of factors that investors might think affect security returns. If the analyst can estimate values for these common factors or indices that affect security returns, he should be able to estimate security returns as well as variances and covariances. The m-index model can be used to estimate time t returns for security i, expected returns, variances and covariances between returns on security i and j as follows:

(1) $$R_{i,t} = \alpha_i + \beta_{i,1} I_{t,1} + \beta_{i,2} I_{t,2} + \ldots.. + \beta_{i,m} I_{t,m} + \varepsilon_{i,t}$$

(2) $$E[R_i] = \alpha_i + \beta_{i,1} E[I_i] + \beta_{i,2} E[I_2] + \ldots + \beta_{i,m} E[I_m]$$

(3) $$\sigma_i^2 = \beta_{i,1}^2 \sigma_{I(1)}^2 + \beta_{i,2}^2 \sigma_{I(2)}^2 + \ldots.. + \beta_{i,m}^2 \sigma_{I(m)}^2 + \sigma_{\varepsilon,i}^2$$

(4) $$\sigma_{i,j} = \beta_{i,1}\beta_{j,1} \sigma_{I(1)}^2 + \beta_{i,2}\beta_{j,2} \sigma_{I(2)}^2 + \ldots.. + \beta_{i,m}\beta_{j,m} \sigma_{I(m)}^2 + \sigma_{\varepsilon(i),\varepsilon(j)}^2$$

Derivations for these measures of expected return, variance and covariance are identical to those for the Single Index Model after adjusting the original statistical measures for the additional indices.

One important problem from a practical perspective concerns how to obtain indices for the Multiple Index Model. Selection of these indices should be based upon the sources of co-movement among security returns. In the Single Index Model, the index selected is often the market portfolio (index, e.g., S&P 500) return. Potential indices for a Multiple Index Model might include market index returns, interest rates, commodity prices, financial ratios, firm size, and volatility of industry. Any economic or fundamental factor might qualify as an index if it captures a significant portion of the co-movement among security prices.

Illustration 5: Multi-Index Model

In this illustration, we will examine the returns of a stock and its relationship with two indices, which might be the return on the market portfolio (Index 1) and the proportional change in the price of a particular commodity. Stock returns and index values are listed in Table 8.

Year	Return	Index 1	Index 2
2001	0.15	0.1	-0.2
2002	0.25	0.1	-0.02
2003	0.5	0.25	0.5
2004	0.35	0.25	-0.5
2005	-0.27	-0.03	-0.4
2006	-0.3	0.08	-0.5
2007	0.4	0.3	-0.2
2008	-0.28	-0.05	-0.23
2009	-0.1	-0.25	-0.34

2010	0.5	0.15	0.1

Table 8: Stock Returns and Index Values

Regression coefficients for the stock are computed as follows based on an OLS regression: $\alpha_i = \beta_{i,0} = 0.099129287$, $\beta_{i,1} = 1.154299498$ and $\beta_{i,2} = 0.46377789$. If we were to assume 2011 values of .05 for Index 1 and -.05 for Index 2, we can make the following statement about the expected return of the stock:

$$E[R_i] = \alpha_i + \beta_{i,1}E[I_i] + \beta_{i,2}E[I_2] = .099 + 1.154 \cdot .05 + .463 \cdot (-.05) = .13355$$

Variances for the two index values are computed as follows: $\sigma^2_{I(1)} = 0.3298$ and $\sigma^2_{I(2)} = 1.1689$. With this information, we can make the following statement about the stock's variance if we ignore unsystematic variances:

$$\sigma_i^2 \approx \beta_{i,1}^2\sigma_{I(1)}^2 + \beta_{i,2}^2\sigma_{I(2)}^2 = 1.154299498^2 \cdot 0.3298 + 0.46377789^2 \cdot 1.1689 = .690847$$

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EXERCISES

1. An investor wishes to combine Stevenson Company stock and Smith Company stock into a riskless portfolio. The standard deviations associated with returns on these stocks are 10% and 18% respectively. The coefficient of correlation between returns on these two stocks is -1. What must be each of the portfolio weights for the portfolio to be riskless?

2. Assume that the coefficient of correlation between returns on all securities equals zero in a given market. There are an infinite number of securities in this market, all of which have the same standard deviation of returns (assume that it is .5). What would be the portfolio return standard deviation if it included all of these infinite number of securities in equal investment amounts? Why? (Demonstrate your solution mathematically.)

3. Investors have the opportunity to invest in any combination of the 5% riskless asset and the two risky securities given in the table below:

(i)	$E[R_i]$	σ_i	$\sigma_{1,i}$	$\sigma_{2,i}$	$\sigma_{3,i}$
1	.15	.50	.25	.05	0
2	.08	.40	.05	.16	0

a. What are the security weights for the optimal (market) portfolio of risky assets?
b. What are the market portfolio expected return and standard deviation levels?

4. An investor with $100,000 has the opportunity to invest in any combination of the securities given in the table below: He wishes to select the most efficient portfolio of these four assets such that his portfolio standard deviation equals .10.

SECURITY(i)	E[R]	COV(i,1)	COV(i,2)	COV(i,3)	COV(i,4)
1	.05	0	0	0	0
2	.10	0	.10	.01	.01
3	.15	0	.01	.20	0
4	.20	0	.01	0	.30

a. How much should the investor invest in each security?
b. Assuming that the four securities above are the only ones available in the market, what is the Beta of Security 3?

5. An investor has the opportunity to invest in a portfolio combining the following two risky stocks:

Security	Expected Return	Standard Deviation	
A	.08	.30	COV(A,B) = 0
B	.12	.60	

The investor can borrow money at a rate of 6%, lend money at 4% and has $500,000 to invest. The investor intends to minimize the risk of her portfolio but requires an expected return of 18%.

How much money should she borrow or lend? How much should she invest in each of the two stocks?

6. Investors have the opportunity to invest in any combination of the securities given in the table below:

(i)	$E[R_i]$	σ_i	$\sigma_{1,i}$	$\sigma_{2,i}$	$\sigma_{3,i}$
1	.25	.40	.16	.05	0
2	.15	.20	.05	.04	0
3	.05	0	0	0	0

Find the slope of the Capital Market Line.

7. Investors have the opportunity to invest in varying combinations of riskless treasury bills and the market portfolio. Investors' investment portfolios will have expected returns equal to $[R_p]$ and standard deviations of returns equal to σ_p. Let w_m be the proportion of a particular investor's wealth invested in the market portfolio. Obviously, the investor's proportional investment in the riskless asset is $w_f = (1-w_m)$. Prove (or derive) the following:

a. $\sigma_P = w_m \sigma_m$

b. $E[R_p] = r_f + \frac{\sigma_p}{\sigma_m}(E[R_m] - r_f)$

Note: If you successfully complete parts a and b, you have derived the equation for the Capital Market Line (where the market portfolio characteristics are known). Now, complete part c:

c. What happens to the slope of the Capital Market Line as each investor's level of risk aversion increases?

8. How would you expect transactions costs to affect borrowing rates of interest? How would lending rates of interest be affected? How would the Capital Market Line be affected by transactions costs?

9. A stock currently selling for $60 has a historical standard deviation of .25 and a coefficient of correlation with the market portfolio of .4. Over the same period, the historical standard deviation of the market portfolio was .16. Investors anticipate dividends of $2 per share in one year at which time the stock can be sold for $65. Determine the following for the stock if the current Treasury Bill rate is .05 and the required return on the market portfolio is .12:

a. The stock Beta.
b. The required return of the stock.
c. The discount rate to be associated with cash flows from the stock.
d. The present value of cash flows associated with the stock.
e. Whether the stock constitutes a good investment.

10. Historical returns for Holmes Company stock, Warren Company stock and the market portfolio along with Treasury Bill (T-Bill) rates are summarized in the following chart:

Year	Holmes	Warren	Market	r_m-r_f	T-Bill

1986	12%	.04	10%	.04	6%
1987	18%	.20	14%	.08	6%
1988	7 %	.02	6 %	0	6%
1989	3 %	-.03	2 %	-.04	6%
1990	10%	.09	8 %	.02	6%

a. Calculate return standard deviations for each of the stocks and the market portfolio.

b. Calculate correlation coefficients between returns on each of the stocks and returns on the market portfolio.

c. Prepare graphs for each of the stocks with axes (R_{it} - R_{ft}) and (R_{mt} - R_{ft}) where R_{it} is the historical return in year (t) for stock (i) ; R_{mt} and R_{mt} are historical market and risk-free returns in time (t). Plot Characteristic Lines for each of the two stocks.

d. Calculate Betas for each of the stocks. How do your Betas compare to the slopes of the stock Characteristic Lines?

11. Consider the Holmes and Warren stocks whose historical returns are given above. Assume an investor had combined each of the stocks into a portfolio such that half of his wealth was invested in each of the stocks. Calculate the following for the investor's portfolio:

a. Historical portfolio standard deviation for the five year period.

b. Historical correlation coefficient between the market portfolio and the investor's portfolio.

c. The portfolio Beta.

d. How does this portfolio Beta compare to the Betas of the individual stocks?

12. Historical returns for the Ripco Fund and the market portfolio along with Treasury Bill (T-Bill) rates (r_f) are summarized in the following table:

Year	Ripco	Market	T-Bill	Year	Ripco	Market	T-Bill
1	0.35	0.28	0.05	11	-0.07	-0.15	0.05
2	0.04	0.05	0.05	12	-0.11	-0.21	0.05
3	0.10	0.09	0.05	13	0.42	0.26	0.05
4	-0.01	-0.02	0.05	14	0.01	0.04	0.05
5	0.38	0.32	0.05	15	0.05	0.08	0.05
6	0.31	0.25	0.05	16	0.07	0.11	0.05
7	0.33	0.26	0.05	17	-0.01	-0.03	0.05
8	0.42	0.30	0.05	18	-0.12	-0.37	0.05
9	0.07	0.14	0.05	19	0.33	0.30	0.05
10	-0.02	-0.06	0.05	20	0.05	0.13	0.05

a. Calculate the fund beta over the 20-year period.

b. Calculate the fund alpha over the 20-year period. Did the fund outperform the market during this period on a risk-adjusted basis?

13. Which of the following vectors are orthogonal to a vector of ones?

$$\vec{a} = \begin{bmatrix} 1 \\ 9 \\ -6 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} -6 \\ 10 \\ -4 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} -1 \\ -4 \\ 5 \end{bmatrix}$$

14. Suppose that a given portfolio A has an expected return of .08, beta with index 1 equal to 1.5 and a beta of 2 with respect to the second index. Portfolio B has an expected return of .18, has a beta of 3 with respect to the first index and a beta of 2.5 with respect to the second index. Portfolio C has an expected return of .08 and has betas equal to one with each of the two indices. What are the three-index expected values implied by the portfolio expected returns?

15. An analyst has concluded that security returns are related to oil prices and corn prices along with the 5% Treasury bill rate. However, he feels that oil prices and corn prices are uncorrelated. He has determined that the Beta of Security A with oil prices is .03 and the Beta of Security A with corn prices is .4. If oil prices are projected to be 1.25 and corn prices are projected to be .18, what return would he forecast for Security A?

SOLUTIONS

1. Here, we want to find that w_A value that will set portfolio variance equal to zero. Remember that portfolio weights must sum to one. Thus, w_B is simply 1 - w_A.

$$\sigma_P^2 = w_A^2 \cdot .10^2 + w_B^2 + .18^2 + 2 \cdot W_A \cdot W_B \cdot .10 \cdot .18 \cdot -1 = 0$$

$$0 = .01w_A^2 + .0324 \cdot (1 - w_A)^2 - .036 \cdot w_A \cdot (1 - w_A)$$

$$0 = .01w_A^2 + .0324 + .324{w_A}^2 - .0648w_A - .036w_A + .036w_A^2$$

$$0 = .0784w_A^2 - .1008w_A + .0324$$

Solve for w_A using the quadratic formula:

$$w_A = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \text{where a=.0784, b=-.1008 and c=.0324.}$$

Plugging in for a, b and c, we find that the portfolio is riskless when w_A = .64286. Thus, w_B = .35714.

Riskless portfolios can be constructed from risky securities only when their returns are perfectly inversely correlated. Even in this case, only one combination of weights results in a riskless portfolio.

2. This would be a perfectly diversified portfolio; its standard deviation will be zero. Portfolio variance is determined as follows:

$$\sigma_P^2 = 2[\sum_{i=1}^{\infty} \sum_{j=1, j \neq i}^{\infty} (1/n \cdot 1/(n-1) \cdot 0] + \sum_{j=1}^{\infty} (1/n)^2 \cdot \sigma_i^2$$

$$\sigma_P^2 = 2[\sum_{i=1}^{\infty} \sum_{j=1, j \neq i}^{\infty} (1/\infty \cdot 1/(\infty-1) \cdot 0] + \sum_{j=1}^{\infty} (1/\infty)^2 \cdot \sigma_i^2 = 0 + 1/\infty \cdot \sigma_i^2 = 0 + 0 = 0$$

3. First, we solve the following linear system for z(1) and z(2):

.25z(1) + .05z(2) = (.15-.05)
.05z(1) + .16z(2) = (.08-.05)

z(1) = .38666667 ; z(2) = .06666667

a. Thus, w(1) = .852941 and w(2) = .147049
b. E[R(m)] = .139706; σ^2_m = .197881; σ_m = .444838

4. For part a, first, note that Security 1 is the riskless asset. We will first find the weights of the optimal portfolio of stocks and then find the optimal combination of stocks and bonds. We begin by solving the following for z(2), z(3) and z(4):

.10z(2) + .01z(3) + .01z(4) = (.10 -.05)
.01z(2) + .20z(3) + 0 z(4) = (.15 -.05)
.01z(2) + 0 z(3) + .30z(4) = (.20 - .05)

We find that z(2) = 0.403361, z(3) = 0.479832 and z(4) = 0.486555. This implies that for the optimal portfolio of stocks (the market portfolio), w(2) = 0.294479, w(3) = 0.350307 and w(4) = 0.355215. Now, the investor needs to determine the optimal mix of stocks and bonds. First, compute the standard deviation of the three-stock portfolio:

$$\sigma_m = \left[w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_{1,2} + 2w_1w_3\sigma_{1,3} + 2w_2w_3\sigma_{2,3}\right]^{.5} = .274268$$

Any portfolio on the Capital Market Line must be a linear combination of riskless bonds and the market portfolio of stocks. Our target portfolio with a standard deviation equal to .10 must be a linear combination of the standard deviations of the riskless asset and of the market portfolio of stocks:

$$\sigma_P = .10 = \left[w_{rf}^2\sigma_{rf}^2 + w_m^2\sigma_m^2 + 2w_{rf}w_m\sigma_{rf,m}\right]^{.5} = \left[0 + w_m^2 \cdot .2274268^2 + 0\right]^{.5}$$

$$w_m = .10/.274268 = .364606$$

Thus, 36.4606% of the investor's money should be invested in the stock portfolio and the remaining $63,539.4 in riskless bonds. Thus, of the $36,460.6 in stocks, $10,736.9 should be invested in Stock 2, $12,772.4 in Stock 3 and $12,951.4 in Stock 4. For part b (which is a bit more difficult still), we know from the derivation of the Capital Asset Pricing Model that:

$$\sigma_{3,m} = (w_2\sigma_{2,3} + w_3\sigma_3^2 + w_4\sigma_{3,4})$$

This implies that: $\sigma_{3,m}$ = .294479*.01 + .350307*.2 + 355215*0 = .0730187

Since the variance of the market is its standard deviation squared = .075223 and the beta of an asset equals its covariance with the market divided by the market variance, the beta of Security 3 is determined to be:

$$\beta_3 = \sigma_{3,m} \div \sigma_m^2 = .0730187/.075223 = .97069$$

5. This problem is complicated by having different borrowing and lending rates. This essentially means that there will be two "Capital Market Lines," one for lending and one for borrowing. Notice that the investor's 18% required return exceeds the return of any of the three securities. This means that the investor will probably need to leverage up her portfolio by borrowing in order to meet her requirement for expected return. Risky asset portfolio characteristics are found from the following:

$.02 = .09z_1 + 0z_2$; $z_1 = .2222$
$.06 = 0z_1 + .36z_2$; $z_2 = .1667$; $w_1=.571$; $w_2=.429$
$E[R_m] = .097$; $w_f=(1-w_m)$; $E[R_p] = .18 = (1-w_m).06 + w_m.097$
$.18 = .06 + w_m.037$; $w_m = 3.243$; Borrow $1,121,500
Invest $1,621,500 in the market - $925,876.5 in security 1 and $695,623.5 in Security 2

$.02 = .09z_1 + 0z_2$; $z_1 = .2222$
$.06 = 0z_1 + .36z_2$; $z_2 = .1667$; $w_1=.571$; $w_2=.429$

$E[R_m] = .097$; $w_f=(1-w_m)$; $E[R_p] = .18 = (1-w_m).06 + w_m.097$

Now, the allocations of funds to the stock portfolio and to the bonds are made:
$.18 = .06 + w_m \times .037$; $w_m = 3.243$; $1-w_m = w_f = -2.243$; Borrow $1,121,500
Invest $1,621,500 in the market: $925,876.5 in security 1 and $695,623.5 in Security 2.

6. Solve the following for z(1) and z(2):
.16z(1) + .05z(2) = (.25-.05)
.05z(1) + .04z(2) = (.15-.05)

z(1) = .7692308 ; z(2) = 1.5384616

Thus, w(1) = .333333333 and w(2) = .666666667

E[R(m)] = .1833 ; σ^2_m = .0577 ; σ_m = .2403 ; Θ = .5547

7.a. $$\sigma_P = \sqrt{w_F^2\sigma_F^2 + w_M^2\sigma_M^2 + 2w_F w_M \sigma_{F,M}}$$

$$\sigma_P = \sqrt{w_F^2 \cdot 0 + w_M^2\sigma_M^2 + 2w_F w_M \cdot 0} = w_M \sigma_M$$

b. $$E[R_P] = w_f E[r_f] + w_m E[R_m]; w_m = \sigma_p / \sigma_m ; w_f = 1 - w_m$$

$$E[R_P] = (1 - \frac{\sigma_p}{\sigma_m} \cdot r_f + \frac{\sigma_p}{\sigma_m} E[R_m]); r_f + \frac{\sigma_p}{\sigma_m}(E[R_m - r_f])$$

c. If the derived CML was for a single investor, an increase in risk aversion will lead to an increased required risk-premium on the market (R_m), increasing the slope of the CML. If the derived CML was for a market of many investors, no single investor will be able to affect its slope. In this case, the slope will remain unchanged; an investor will simply vary his holdings of the riskless asset.

8. Increase the cost of borrowing: if the borrower pays this cost, borrowing rates increase. Decrease the return from lending: lending rates decrease if the lender pays this cost. In summary, transactions costs increase interest rates; get two separate lines, one for borrowing and one for lending.

9. Given: $P_0 = 60 \sigma_a = .25$

$P_1 = 65$ $\sigma_m = .16$

$\sigma_{am} = .40$ $R_f = .05$

Div= \$2 $R_m = .12$

$cov_{am} = \sigma_a\sigma_m\rho_{am}$ = .25(.16)(.4) = .016

a. $\beta = cov_{am}/\sigma_m^2$ = .016/(.16)^2 = .016/.0256 = .625

b. $rr_a = R_f + \beta_a(R_m - R_f)$ = .05 + .625(.12 - .05) = .05 + .625(.07) = .09375

c. same as b = .09375

d. PV = $CF_a/(1+rr_a)$ = (65 + 2)/(1 + .09375) = 67/1.09375 = 61.257

e. The stock is a good buy because PV > P_0; \$61.257 > \$60.

. 10.a. Calculate the variance of risk premiums for market portfolio:

$$\sigma_m^2 = \frac{(.04-.02)^2 + (.08-.02)^2 + (0-.02)^2 + (-.04-.02)^2 + (.02-.02)^2}{5-1} = .008/4 = .002$$

b. Calculate the covariance between risk premiums on the stock and risk premiums on the market portfolio:

$$\sigma_{H,m} = \sum_{t=1}^{m} \frac{(R_{H,t} - E(R_{H,t})) \cdot (R_{m,t} - E(R_{m,t}))}{n-1}$$

{[(.06 -.04) * (.04 -.02) + (.12 -.04) * (.08 -.02) + (.01 -.04) * (0 -.02) + (-.03-.04) * (-.04 -.02)+ (.04 -.04) *(.02 -.02)] /(5-1)

$$= \frac{(.02)\cdot(.02)+(.08)\cdot(.06)+(-.03)\cdot(-.02)+(-.07)\cdot(-.06)+0}{4} = .0025$$

b. Calculate beta for the stock by dividing the covariance between the stock and market risk premiums by the variance of premiums on the market portfolio:

$$\beta_H = \frac{\sigma_{H,m}}{\sigma_m^2} = \frac{.0025}{.002} = 1.25$$

$$\beta_w = \sigma_w \sigma_m \rho_{w,m} \;/ \sigma_m^2 = (.078)(.04)(.909)/(.04)^2 \;;\; \beta_m = 1$$

The betas are the slopes of the characteristic lines.

d. See part c for Beta calculations.

$$\beta_h = \sigma_h \sigma_m \rho_{hm} / \sigma_m^2 = (.0502)(.04)(.996)/(.04)^2 = .0548/.04 = 1.25$$

$$\beta_w = \sigma_w \sigma_m \rho_{wm} / \sigma_m^2 = .078(.04)(.909)/(.04)^2 = 1.775 \qquad B_m = 1$$

The betas are the slopes of the characteristic lines.

11. a. First, calculate return standard deviations for each of the stocks and the market portfolio:

$$\overline{R}_h = \sum_{t=1}^{5} R_{ht} \div n = (.12+.18+.07+.03+.10)/5 = .10$$

$$\overline{R}_w = \sum_{t=1}^{5} R_{wt} \div n = (.04+.20+.02+.03+.09)/5 = .064$$

$$\overline{R}_m = \sum_{t=1}^{5} R_{mt} \div n = (.10+.14+.06+.02+.08)/5 = .08$$

$$\sigma_h = \left[\frac{(.12-.10)^2+(.18-.10)^2+(.07-.10)^2+(.03-.10)^2+(.10-.10)^2}{5}\right]^{1/2} = .50199$$

$$\sigma_w = \left[\frac{(.04-.064)^2+(.2-.064)^2+(.02-.064)^2+(-.03-.064)^2+(.09-.064)^2}{5}\right]^{1/2} = .078128$$

$$\sigma_m = \left[\frac{(.10-.08)^2+(.14-.08)^2+(.06-.08)^2+(.02-.08)^2+(.08-.08)^2}{5}\right]^{1/2} = .04$$

b. Calculate correlation coefficients between returns on each of the stocks and returns on the market portfolio.

$\sigma_{h,m} = \sum_{t=1}^{5}(R_{h,t}-E(R_{h,t}))(R_{m,t}-E(R_{m,t}))\div n$ =[{(.12-.10) (.10-.08) + (.18-.10) (.14-.08) + (.07-.10) (.06-.08)} + {(.03-.10) (.02-.08) + (.10-.10) (.08 - .08)}] / {5}

$$= \frac{(.02)(.02)+(.08)(.06)+(-.03)(-.02)+(-.07)(-.06)+0}{5} = .002$$

$$\sigma_{h,w} = \sum_{t=1}^{5}(R_{h,t}-E(R_{h,t}))(R_{w,t}-E(R_{w,t}))\div n$$

$$=[\{(.12-.10)(.04-.064)+(.18-.10)(.20-.064)+(.07-.10)(.02-.064)\} + \{(.03-.10)(-.03-.064)+(.10-.10)(.09-.064)\}] / \{5\}$$

$$=\frac{(.02)(-.024)+(.08)(.0136)+(-.03)(-.044)+(-.07)(-.094)+0}{5}=.00366$$

$$\sigma_{w,m}=\sum_{t=1}^{5}(R_{h,t}-E(R_{h,t}))(R_{w,t}-E(R_{w,t}))\div n$$

$$=[\{(.04-.064)(.10-.08)+(.14-.08)(.20-.064)+(.06-.08)(.02-.064)\}+\{(.02-.08)(-.03-.064)+(.08-.08)(.09-.064)\}]/5=.00284$$

$$\rho_{h,m}=Cov_{h,m}/\sigma_h\sigma_m=\sigma_{h,m}/\sigma_h\sigma_m = .002/\{(.0502)(.04)\}= .996$$

$$\rho_{w,m}=Cov_{w,m}/\sigma_w\sigma_m=\sigma_{w,m}/\sigma_w\sigma_m = .00284/\{(.078)(.04)\}= .909$$

c. The slopes of the lines are the stock Betas.

d. Calculate Betas for each of the stocks. How do your betas compare to the slopes of the stock characteristic lines?

$$\beta_h=\sigma_h\sigma_m\rho_{h,m}/\sigma_m^2 = (.0502)(.04)(.996)/(.04)^2 = .0548/.04 = 1.25$$

$$\beta_h=\sigma_w\sigma_m\rho_{w,m}/\sigma_m^2 = (.078)(.04)(.909)/(.04)^2 = 1.775 \qquad \beta_m = 1$$

The betas are the slopes of the characteristic lines.

12.a. First, calculate risk premiums for both the fund and the market as follows:

t	R_p	R_m	r_f	R_p-r_f	R_m-r_f
1	0.35	0.28	0.05	0.3	0.23
2	0.04	0.05	0.05	-0.01	0
3	0.1	0.09	0.05	0.05	0.04
4	-0.01	-0.02	0.05	-0.06	-0.07
5	0.38	0.32	0.05	0.33	0.27
6	0.31	0.25	0.05	0.26	0.2
7	0.33	0.26	0.05	0.28	0.21
8	0.42	0.3	0.05	0.37	0.25
9	0.07	0.14	0.05	0.02	0.09
10	-0.02	-0.06	0.05	-0.07	-0.11
11	-0.07	-0.15	0.05	-0.12	-0.2
12	-0.11	-0.21	0.05	-0.16	-0.26
13	0.42	0.26	0.05	0.37	0.21
14	0.01	0.04	0.05	-0.04	-0.01
15	0.05	0.08	0.05	0	0.03
16	0.07	0.11	0.05	0.02	0.06
17	-0.01	-0.03	0.05	-0.06	-0.08
18	-0.12	-0.37	0.05	-0.17	-0.42
19	0.33	0.3	0.05	0.28	0.25
20	0.05	0.13	0.05	0	0.08

Next, run a simple ordinary least squares regression of (R_p-r_f) on (R_m-r_f). The fund's beta is the slope term in the regression and the fund's alpha is its vertical intercept:

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.910030353
R Square	0.828155243
Adjusted R Square	0.818608312
Standard Error	0.079179681
Observations	20

ANOVA

	df	SS	MS	F	Significance F
Regression	1	0.543845407	0.543845407	86.74570346	2.63657E-08
Residual	18	0.112849593	0.006269422		
Total	19	0.656695			

	Coefficients	Standard Error	t Stat	P-value
Intercept	0.045004834	0.018088349	2.488056431	0.022868369
X Variable 1	0.895978331	0.096199656	9.313737352	2.63657E-08

Thus, the fund beta is .8959.
b. The fund alpha is .04. Since the alpha is statistically significant at the 5% level, we can conclude that the fund has outperformed the market.

13. Multiply vectors as follows:

$$\begin{bmatrix}1 & 9 & -6\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=4 \qquad \begin{bmatrix}-6 & 10 & -4\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=0 \qquad \begin{bmatrix}-1 & -4 & 5\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=0$$

$$\mathbf{a}^T \quad \mathbf{i} = 4 \qquad \mathbf{b}^T \quad \mathbf{i} = 0 \qquad \mathbf{c}^T \quad \mathbf{i} = 0$$

Vectors **b** and **c** are orthogonal to unit vectors.

14. We can write the expected returns vector **E[r]**, the beta matrix **β**, the unit vector **i** and the factor risk premia vector **δ** as follows:

$$E[\vec{r}]=\begin{bmatrix}.08\\.18\\.08\end{bmatrix} \quad [\beta]=\begin{bmatrix}1.5 & 2\\3 & 2.5\\1 & 1\end{bmatrix} \quad \vec{i}=\begin{bmatrix}1\\1\\1\end{bmatrix} \quad \delta=\begin{bmatrix}\delta_1\\\delta_2\end{bmatrix}$$

The expected returns vector is expressed as follows:

$$E[\bar{r}] = \begin{bmatrix} .08 \\ .18 \\ .08 \end{bmatrix} = \delta_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1.5 & 2 \\ 3 & 2.5 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

We can solve the above for δ_0, δ_1 and δ_2 to obtain factor risk premia. This system can also be expressed as a system of equations as follows:

$$\begin{aligned} 0.08 &= 1\delta_0 + 1.5\delta_1 + 2\delta_2 \\ 0.18 &= 1\delta_0 + 3\delta_1 + 2.5\delta_2 \\ 0.08 &= 1\delta_0 + 1\delta_1 + 1\delta_2 \end{aligned}$$

In matrix format, our system of equations is as follows:

$$\begin{bmatrix} 1 & 1.5 & 2 \\ 1 & 3 & 2.5 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} .08 \\ .18 \\ .08 \end{bmatrix}$$

We invert the coefficients (betas) matrix and solve for **δ** as follows:

$$\begin{bmatrix} -4 & -4 & 1.8 \\ -1.2 & 0.8 & 0.4 \\ 1.6 & -0.4 & -1.2 \end{bmatrix} \begin{bmatrix} .08 \\ .18 \\ .08 \end{bmatrix} = \begin{bmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} .04 \\ .08 \\ -.04 \end{bmatrix}$$

The riskless return rate and two factor risk premia are determined to be as follows**:** $\delta_0 = .04 = r_f$, $\delta_1 = .08$, $\delta_2 = -.04$.

15. $E[R_a] = r_f + w_{ao}(E(R_o)) + w_{ac}(E(R_c)) = .05 + .03(1.25) + .4(.18) = .05 + .0375 + .072 = .1595$

APPENDIX A: SOLVING SYSTEMS OF EQUATIONS WITH MATRICES

Multiplication of Matrices

Two matrices **A** and **B** may be multiplied to obtain the product **AB** = **C** if the number of columns in the first Matrix **A** equal the number of rows **B** in the second.[1] If Matrix **A** is of dimension m × n and Matrix **B** is of dimension n × q, the dimensions of the product Matrix **C** will be m × q. Each element $c_{i,k}$ of Matrix **C** is determined by the following sum:

$$c_{i,k} = \sum_{j=1}^{n} a_{i,j} b_{j,k}$$

For example, consider the following product:

$$\begin{bmatrix} 7 & 4 & 9 \\ 6 & 4 & 12 \\ 3 & 2 & 17 \end{bmatrix} \begin{bmatrix} 7 & 6 \\ 5 & 1 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 150 & 154 \\ 170 & 184 \\ 184 & 224 \end{bmatrix}$$

A **.** **B** **=** **C**

Matrix **C** in the above is found as follows:

$$\begin{bmatrix} 7 & 4 & 9 \\ 6 & 4 & 12 \\ 3 & 2 & 17 \end{bmatrix} \begin{bmatrix} 7 & 6 \\ 5 & 1 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} (7\cdot 7)+(4\cdot 5)+(9\cdot 9) & (7\cdot 6)+(4\cdot 1)+(9\cdot 12) \\ (6\cdot 7)+(4\cdot 5)+(12\cdot 9) & (6\cdot 6)+(4\cdot 1)+(12\cdot 12) \\ (3\cdot 7)+(2\cdot 5)+(17\cdot 9) & (3\cdot 6)+(2\cdot 1)+(17\cdot 12) \end{bmatrix}$$

A **.** **B** **=** **C**

Notice that the number of columns (3) in Matrix **A** equals the number of rows in Matrix **B**. Also note that the number of rows in Matrix **C** equals the number of rows in Matrix **A**; the number of columns in **C** equals the number of columns in Matrix **B**.

Inverting Matrices

An *inverse* Matrix $\mathbf{A}^{-1}$ exists for the square Matrix **A** if the product $\mathbf{A}^{-1}\mathbf{A}$ or $\mathbf{A}\mathbf{A}^{-1}$ equals the identity Matrix **I**. Consider the following product:

$$\begin{bmatrix} 2 & 4 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{30} & \frac{2}{15} \\ \frac{4}{15} & \frac{-1}{15} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A $\mathbf{A}^{-1}$ **=** **I**

One means for finding the inverse Matrix $\mathbf{A}^{-1}$ for Matrix **A** is through the use of a process called the *Gauss-Jordan Method*. This method will be performed on Matrix **A** by first augmenting it with the identity matrix as follows:

(A) $$\begin{bmatrix} 2 & 4 & \vdots & 1 & 0 \\ 8 & 1 & \vdots & 0 & 1 \end{bmatrix}$$

For the sake of convenience, call the above augmented Matrix **A** temporarily. Now, a series of row operations (addition, subtraction or multiplication of each element in a row) will be performed such that the identity matrix replaces the original Matrix **A** (on the left side). The right-side elements will comprise the inverse Matrix $\mathbf{A}^{-1}$. Thus, in our final augmented matrix, we will have ones along the principal diagonal on the left side and zeros elsewhere; the right side of the matrix will comprise the inverse of **A**. Allowable row operations include the following:

1. Multiply a given row by any constant. Each element in the row must be multiplied by the same constant.
2. Add a given row to any other row in the matrix. Each element in a row is added to the corresponding element in the same column of another row.
3. Subtract a given row from any other row in the matrix. Each element in a row is subtracted from the corresponding element in the same column of another row.
4. Any combination of the above. For example, a row may be multiplied by a constant before it is subtracted from another row.

Our first row operation will serve to replace the upper left corner value with a one. We multiply Row 1 in **A** (Row 1**A**) by .5 to obtain the following:

$$\begin{bmatrix} 1 & 2 & \vdots & .5 & 0 \\ 8 & 1 & \vdots & 0 & 1 \end{bmatrix} \quad \begin{matrix} 1A \cdot .5 = 1B \\ 2A \end{matrix}$$

where Row 1**B** replaces Row 1**A**. Now we obtain a zero in the lower left corner by multiplying Row 2 in **A** by 1/8 and subtracting the result from our new Row 1 to obtain Matrix **B** as follows:

(B) $$\begin{bmatrix} 1 & 2 & \vdots & .5 & 0 \\ 0 & \frac{15}{8} & \vdots & .5 & \frac{-1}{8} \end{bmatrix} \quad \begin{matrix} 1A \cdot .5 = 1B \\ 1B - (2A \cdot \frac{1}{8}) = 2B \end{matrix}$$

Next, we obtain a 1 in the lower right corner of the left side of the matrix by multiplying Row 2**B** by 8/15:

$$\begin{bmatrix} 1 & 2 & \vdots & .5 & 0 \\ 0 & 1 & \vdots & \frac{4}{15} & \frac{-1}{15} \end{bmatrix} \quad \begin{matrix} 1A \cdot .5 = 1B \\ 2B \cdot \frac{8}{15} = 2C \end{matrix}$$

We obtain a zero in the upper right corner of the left side matrix by multiplying Row 2 above by 2 and subtracting from Row 1 in **B**:

(C) $\begin{bmatrix} 1 & 0 & \vdots & \frac{-1}{30} & \frac{2}{15} \\ 0 & 1 & \vdots & \frac{4}{15} & \frac{-1}{15} \end{bmatrix}$ $\begin{array}{l} 1B-(2C\cdot 2)=1C \\ 2B\cdot\frac{8}{15}=2C \end{array}$

The left side of augmented Matrix **C** is the identity matrix; the right side of **C** is $\mathbf{A}^{-1}$.

Because matrices cannot be divided as numbers are in arithmetic, one performs an analogous operation by inverting the matrix intended to be the "divisor" and post multiplying this inverse by the first matrix to obtain a quotient. Thus, instead of dividing **A** by **B** to obtain **D**, one inverts **B** and obtains **D** by the product $\mathbf{AB}^{-1} = \mathbf{D}$. This concept is extremely useful for many types of algebraic manipulations.

Solving Systems of Equations

Matrices can be very useful in arranging systems of equations. Consider for example the following system of equations:

$$.05x_1 + .12x_2 = .05$$
$$.10x_1 + .30x_2 = .08$$

This system of equations may be represented as follows:

$$\begin{bmatrix} .05 & .12 \\ .10 & .30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .05 \\ .08 \end{bmatrix}$$

$$\mathbf{C} \quad \mathbf{x} \quad = \quad \mathbf{s}$$

We are not able to divide **C** by **s** to obtain **x**; instead, we invert **C** to obtain $\mathbf{C}^{-1}$ and multiply it by **s** to obtain **x**:

$$\mathbf{C}^{-1}\mathbf{s} = \mathbf{x}$$

Therefore, to solve for Vector **x**, we first invert **C** by augmenting it with the Identity Matrix:

(A) $\begin{bmatrix} .05 & .12 & \vdots & 1 & 0 \\ .10 & .30 & \vdots & 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 2.4 & \vdots & 20 & 0 \\ 0 & .6 & \vdots & -20 & 10 \end{bmatrix}$ $\begin{array}{l} RowB1 = A1\cdot 20 \\ RowB2 = (10\cdot A2) - B1 \end{array}$

(C) $\begin{bmatrix} 1 & 0 & \vdots & 100 & -40 \\ 0 & 1 & \vdots & \frac{-100}{3} & \frac{50}{3} \end{bmatrix}$ $\begin{array}{l} RowC1 = B1-(2.4\cdot C2) \\ RowC2 = B2\cdot 5/3 \end{array}$

$$\mathbf{I} \qquad \mathbf{C^{-1}}$$

Thus, we obtain Vector **x** with the following product:

$$\text{(D)} \quad \begin{bmatrix} 100 & -40 \\ \dfrac{-100}{3} & \dfrac{50}{3} \end{bmatrix} \begin{bmatrix} .05 \\ .08 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.8 \\ \dfrac{-1}{3} \end{bmatrix}$$

$$\mathbf{C^{-1}} \qquad \mathbf{s} \quad = \quad \mathbf{x} \quad = \quad \mathbf{x}$$

Thus, we find that $x_1 = 1.8$ and $x_2 = -1/3$.

Arbitrage is defined as the simultaneous purchase and sale of assets or portfolios yielding identical cash flows. Assets generating identical cash flows (certain or risky cash flows) should be worth the same amount. This is known as the Law of One Price. If assets generating identical cash flows sell at different prices, opportunities exist to create a profit by buying the cheaper asset and selling the more expensive asset. The ability to realize a profit from this type of transaction is known as an arbitrage opportunity. Solutions for multiple variables in systems of equations are most useful in the application of the Law of One Price and seeking arbitrage opportunity.

13.a. First, note that for any security pair *i, j* where i = j, $\sigma_{i,j} = \sigma_i^2$.

$$L = \sigma_A^2 x_A^2 + \sigma_B^2 x_B^2 + \sigma_C^2 x_C^2 + 2\sigma_{A,B}x_A x_B + 2\sigma_{A,C}x_A x_C + 2\sigma_{B,C}x_B x_C$$
$$+ \lambda_1(r_p - r_1x_1 - r_2x_2 - r_3x_3) + \lambda_2(1 - x_1 - x_2 - x_3)$$

$$L = .04x_A^2 + .16x_B^2 + .64x_C^2 + .08x_A x_B + 0x_A x_C + 0x_B x_C$$
$$+ \lambda_1(.09 - .05x_1 - .15x_2 - .25x_3) + \lambda_2(1 - x_1 - x_2 - x_3)$$

b. The partial derivative of L with respect to x_A is written as follows (Note that $\sigma_{A,A} = \sigma_A^2$:

$$\frac{\partial L}{\partial x_A} = 2\sigma_A^2 x_A + 2\sigma_{A,B}x_B + 2\sigma_{A,C}x_C - E[r_i]\lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_A} = .08x_A + .08x_B + 0x_C - .05\lambda_1 - \lambda_2 = 0$$

c. Finding partial derivatives of L with respect each of 5 unknowns, setting each partial derivative equal to 0 and rearranging leads to the following system:

$$\begin{bmatrix} .08 & .08 & 0 & -.05 & -1 \\ .08 & .32 & 0 & -.15 & -1 \\ 0 & 0 & 1.28 & -.25 & -1 \\ -.05 & -.15 & -.25 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -.09 \\ -1 \end{bmatrix}$$

$$\mathbf{V^*} \qquad \mathbf{w^*} \; = \; \mathbf{o^*}$$

We invert Matrix **V*** then solve for Vector **w***. We find the following weights: $x_1 = .717241$, $x_2 = .165517$ and $x_3 = .117241$; our LaGrange multipliers are $\lambda_1 = .397241$ and $\lambda_2 = .050759$.

d. $\sigma_m = \left[x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + x_C^2\sigma_C^2 + 2x_Ax_B\sigma_{A,B} + 2x_Ax_C\sigma_{A,C} + 2x_Bx_C\sigma_{B,C}\right]^{.5} = .207979$

e. We will solve the following for r_f:

$$E[r_m - r_f]\lambda_1 = 2\sigma_m^2$$

$$r_f = -\frac{2\sigma_m^2}{\lambda_1} + E[r_m] = -\frac{2\times .043255}{.397241} + .09 = -.12778$$

Note that this is an odd result with a negative interest rate.

d. We calculate the covariance as follows:

$$2\sigma_{m,i} = E[r_i - r_f]\lambda_1$$

$$\sigma_{m,A} = \frac{E[r_A - r_f]\lambda_1}{2} = \frac{E[.05 + .12778] \times .397241}{2} = .035311$$

e. Since $\beta_i = \frac{\sigma_{m,i}}{\sigma_m^2}$, the beta of security A is $.035311/.207979^2 = .816336$

Appendix B: Adjusted Beta and Index Models

Simple observation of security markets reveals a strong tendency for security returns to be affected by common factors, particularly the market portfolio. From a mathematical perspective, these factors represent a source of covariance or correlation between returns of pairs of securities. The single index model specifies a single source of covariance among security returns $R_{i,t}$, and denotes security returns as a linear function of this factor or index I_t:

(1) $$R_{it} = \alpha_i + \beta_i I_t + \varepsilon_{it}$$

where α_i represents that portion of the return of security i which is constant and independent of the index I_t, β_i represents the sensitivity of security i to index I and ϵ_{it} represents the portion of security i's return which is security specific and unrelated to the index or to returns of other securities. The index models are simply regression models that presume that security returns are a linear function of one or more (in the case of multi-index models) indices. If index models can be used to generate security returns, then the process for obtaining security variances and covariances with respect to one another will be much simplified.

The Single Index Model has several uses:

1. To reduce the number of inputs and computations required for portfolio analysis. In particular, the Single Index Model will be useful for deriving forecasts for security and portfolio expected return, variance and covariance.
2. To build and apply equilibrium models such as the Capital Asset Pricing Model and Arbitrage Pricing Theory.
3. To adjust for risk in event studies and back-testing programs.

The Single Index Model is based on the following series of assumptions:

1. As indicated above, security returns are linear in a common index as follows:
$$R_{it} = \alpha_i + \beta_i I_t + \varepsilon_{it}$$
2. The parameters of the index model α_i and β_i are computed through a linear regression procedure such that the risk premium is purely a function of the index, not security specific risk. That is, $E(\varepsilon_{it})=0$. Furthermore, it will be assumed that security specific risk is unrelated to the value of the index; that is, $E(\epsilon_{it}\beta I_t)=0=Cov(\epsilon_{it}\beta I_t)$.
3. The index represents the only source of covariance between asset returns. That is, $E(\epsilon_{it}\epsilon_{jt})=0$.

Based on the Single Index Model, we may reflect the expected return of a security i or portfolio p as follows:

(2) $$[R_i] = \alpha_i + \beta_i E[I]$$

(2a) $$[R_p] = \alpha_p + \beta_p E[I]$$

where the parameters for the portfolio are simply a weighted average of the parameters for the individual securities. For sake of notational convenience, we use the expectations operator E[*] to replace the summation notation $\Sigma_{i=1}[*]P_i$. That is, for expected security return and variance, we have:

(A) $$[R_i] = \sum_{j=1}^{m}\left[R_i\right]P_i = \alpha_i + \beta_i \sum_{j=1}^{m}\left[I_i P_i\right]$$

(B) $$\sigma_i^2 = E\left[(\beta_i^2 (I - E\left[I\right])^2 + \varepsilon_i^2 + 2\beta_i (I - E\left[I\right])\right]$$

We can use Equations 1 and 2 and our standard definition for security variance to express security variance as a function of the index:

(A) $$\sigma_i^2 = E\left[(\alpha_i + \beta_{i1} + \varepsilon_i) - (\alpha_i + \beta_i E\left[I\right])\right]^2 = E\left[\beta_i (I - E\left[I\right]) + \varepsilon_i\right]^2$$

We can complete the square of Equation A and write security variance as:

(B) $$\sigma_i^2 = E\left[(\beta_i^2 (I - E\left[I\right])^2 + \varepsilon_i^2 + 2\beta_i (I - E\left[I\right])\right]$$

Because the covariance between the index and firm specific returns is assumed to be zero above $(E[(\epsilon_{it}-0)(I-E[I])] = 0)$, the cross product terms drop out:

(C) $$\sigma_i^2 = E\left[(\beta_i^2 (I - E\left[I\right])^2 + \varepsilon_i^2\right] = \beta_i^2 E(I - E\left[I\right])^2 + E(\varepsilon_i - 0)$$

Due to our definition of variance, and that the expected unsystematic risk premium (error) equals zero, Equation C simplifies to:

(3) $$\sigma_i^2 = \beta_i^2 \sigma_I^2 + \sigma_{\varepsilon i}^2$$

This expression has a particularly useful intuition: security variance is the sum of systematic or index induced variance $\beta^2_i\sigma^2_i$ and firm specific variance $\sigma^2_{\epsilon i}$. Firm specific risk $\sigma^2_{\epsilon i}$ will tend towards zero in a well-diversified portfolio such that portfolio variance is expressed:

(3a) $$\sigma_p^2 = \beta_p^2 \sigma_I^2$$

The Single Index Model can be used to substantially reduce the number of computations for covariances required for portfolio risk analysis. We know that n^2 covariance calculations are required to compute portfolio risk.[2] For example, a ninety- security portfolio will require 8100 covariance calculations. Thus, it is very useful to limit the number of calculations required for

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[2]In sum, n^2 covariances need to be computed for the standard portfolio variance model. However, this number of covariances can be reduced to $(n^2-n)/2$ non-trivial covariances since n of the covariances will actually be variances (the covariance between any security i and itself is variance) and each $\sigma_{i,k}$ will equal $\sigma_{k,i}$. By this formula, we can compute that a ninety-security portfolio would require 4005 covariance calculations.

each covariance. Using the expectations operator notation, we can define covariance as follows:

(4) $\sigma_{i,j} = E\big[(R_i - E[R_i])(R_j - E[R_J])\big]$

Replacing Equations 1 and 2 into Equation 4, we have:

(A) $\sigma_{i,j} = Cov(R_i, R_j) = E\big[(\alpha_i + \beta_i I + \varepsilon_i - \alpha_i - \beta_i E[I])(\alpha_j + \beta_j I + \varepsilon_j - \alpha_j - \beta_j E[I])\big]$

After performing multiplications within the brackets, and noting that ϵ_i and ϵ_j are uncorrelated with the index such that the cross-product terms drop out, Equation A simplifies to:

(B) $\sigma_{i,j} = E[\beta_p \beta_j (I - E[I])^2] + \varepsilon_i \varepsilon_j$

We bring the expectations operator inside the brackets to obtain:

(C) $\sigma_{i,j} = \beta_i\, \beta_j E(I - E[I])^2 + E(\varepsilon_i \varepsilon_j)$

Since $(\epsilon_i \epsilon_j)$ equals zero by our assumption above that the index captures all sources of covariance between pairs of securities, Equation C simplifies to:

(5) $\sigma_{i,j} = \beta_i \beta_j \sigma_I^2$

If our covariance calculations were to be based on 60 months of time series returns, we would compute a single beta value for each of n securities in a portfolio and a variance for the index itself. Thus, we could compute all $(n^2\text{-}n)/2$ covariances from n betas and one variance. When n is large, the time to complete these computations will be substantially less than the time to compute $(n^2\text{-}n)/2$ covariances from 60 months of raw returns data.

In most cases, the single index model relies on an index representing market returns. The most frequently used index for academic studies is the S&P 500, but other indices such as those provided by the exchanges, Value Line and Russell may be used as well.

Historical betas are most frequently estimated on the basis of covariances and variance drawn from sixty months of historical security returns. However, historical returns and their volatility are not necessarily the best indicators of future betas. Corporate circumstances change over time as do the markets evaluation of those circumstances. Furthermore, any historical beta estimate would be subject to sampling and measurement error. Blume [1975] has shown a tendency for betas to drift towards 1 over time. He proposed a correction for this tendency to drift towards one:

(6) $\beta_{i,F} = \gamma_0 + \gamma_1 \beta_{i,H}$

where $\beta_{i,F}$ is the forecasted beta for a future five year period and $\beta_{i,H}$ is the historical beta estimated using the procedure described above. The coefficients γ_0 and γ_1 are determined by performing a regression of five-year betas against betas estimated over the immediately preceding five-year period. For example, the beta estimates $\beta_{i,F}$ for the period 1955-1961 based on beta estimates for 1948-1954. $\beta_{i,H}$ were obtained from adjustment coefficients $\gamma_0 = .343$ and $\gamma_1 = .677$. Note that the coefficients will normally sum to approximately one. Other adjustment procedures exist as well, including that proposed by Vasicek [1973].

Beaver, Kettler and Scholes [1970] and numerous papers authored by Barr Rosenberg, including Rosenberg and James [1976], have proposed estimating betas from firm fundamental factors including ratios. The advantage to this methodology is that the "fundamental beta" is not based on historical returns data but on current financial statement data supplemented with other current and relevant information. The fundamental beta forecast $\beta_{i,F}$ is determined as a function of m firm fundamental factors x_i:

(7) $$\beta_{i,F} = \gamma_0 + \gamma_1 x_{i,1} + \gamma_{i,2} + \ldots\ldots + \gamma_{i,m}$$

The fundamental factors might include financial ratios such as debt-equity ratios, liquidity ratios and return measures. Other factors that might be considered relevant include firm size, tenure of C.E.O., volatility of industry, etc. The coefficients are determined on the basis of a regression of historical betas on historical values for the various fundamental factors.

Multi-Index Models

The Multi-Index Model enables the analyst to attribute multiple sources of covariance between security returns. The multi-index model can be used to estimate security returns, expected returns, variances and covariances as follows:

(8) $$R_{it} = \alpha_i + \beta_{i,1} I_{t,1} + \beta_{i,2} I_{t,2} + \ldots\ldots + \beta_{i,m} I_{t,m} + \varepsilon_{i,t}$$

(9) $$E[R_i] = \alpha_i + \beta_{i,1} E[I_i] + \beta_{i,2} E[I_2] + \ldots + \beta_{i,m} E[I_m]$$

(10) $$\sigma_i^2 = \beta_{i,1}^2 \sigma_{I(1)}^2 + \beta_{i,2}^2 \sigma_{I(2)}^2 + \ldots\ldots + \beta_{i,m}^2 \sigma_{I(m)}^2 + \sigma_{\varepsilon i}^2$$

(11) $$\sigma_{i,j} = \beta_{i,1} \sigma_{j,1} + \beta_{i,2}^2 \sigma_{I(2)}^2 + \ldots\ldots + \beta_{i,m}^2 \sigma_{I(m)}^2 + \sigma_{\varepsilon i}^2$$

Derivations for these measures are identical to those for the Single Index Model after adjusting the original statistical measures for the additional indices.

One important problem from a practical perspective concerns how to obtain indices for the Multiple Index Model. Selection of these indices should be based upon the sources of co-movement among security returns. Potential indices might include market index returns, interest rates, commodity prices, financial ratios, firm size, and volatility of industry. Any economic or fundamental factor might qualify as an index if it captures a significant portion of the co-movement among security prices.