Coursepack Chapter 9: OPTIONS

1. Calls and Puts

As we discussed earlier, derivative securities have payoff functions derived from the values of other securities, rates, or indices. One of the more common types of derivative securities is a stock option, which is a legal contract that grants its owner the right (not obligation) to either buy or sell a given stock. There are two types of stock options: puts and calls. A call grants its owner to purchase stock (called underlying shares) for a specified exercise price (also known as a striking price or exercise price) on or before the expiration date of the contract. In a sense, a call is similar to a coupon that one might find in a newspaper enabling its owner to, for example, purchase a roll of paper towels for one dollar. If the coupon represents a bargain, it will be exercised and the consumer will purchase the paper towels. If the coupon is not worth exercising, it will simply be allowed to expire. The value of the coupon when exercised would be the amount by which value of the paper towels exceeds one dollar (or zero if the paper towels are worth less than one dollar). Similarly, the value of a call option at exercise equals the difference between the underlying market price of the stock and the exercise price of the call.

An investor can take any combination of positions in an underlying security and/or calls and puts that trade on the security. Long positions reflect purchases while short positions reflect sales. Table A.1 describes a single long and short position in each of the six individual securities. Future (time T, expiration date) payoffs are given in the table. Long positions require time zero purchase payments of S_0, c_0 and p_0 to invest, while short positions resulting from sales result in time zero cash flows of S_0, c_0 and p_0 to the sellers.

Your Position	Payoff if $S_T \leq X$	Payoff if $S_T > X$	Notes on your Position
Long Underlying	S_T	S_T	You own the underlying asset
Short Underlying	$- S_T$	$- S_T$	You short sold the underlying asset
Long Call	0	$S_T - X$	You dispose of or exercise the call
Short Call	0	$-(S_T - X)$	You are obliged to allow exercise
Long Put	$(X - S_T)$	0	You exercise or dispose of the put
Short Put	$-(X - S_T)$	0	You are obliged to allow exercise

Table A.1: Stock and Plain Vanilla Option Position Payoffs

Illustration

Suppose, for example, that there is a call option with an exercise price of $90 on one share of stock. The option expires in one year. This share of stock is expected to be worth either $80 or $120 in one year, but we do not know which at the present time. If the stock were to be worth $80 when the call expires, its owner should decline to exercise the call. It would simply not be practical to use the call to purchase stock for $90 (the exercise price) when it can be purchased in the market for $80. The call would expire worthless in this case. If, instead, the stock were to be worth $120 when the call expires, its owner should exercise the call. Its owner would then be able to pay $90 for a share that has a market value of $120, representing a $30 profit. In this case, the call would be worth $30 when it expires. Let T designate the options term to expiry, S_T the stock value at option expiry and c_T be the value of the call option at expiry determined as follows:

(1) $$c_T = MAX[0, S_T - X]$$

When S_T = 80, c_T = MAX[0, 80 – 90] = 0
When S_T=120, c_T = MAX[0, 120 – 90] = 30

A put grants its owner the right to sell the underlying stock at a specified exercise price on or before its expiration date. A put contract is similar to an insurance contract. For example, an owner of stock may purchase a put contract ensuring that he can sell his stock for the exercise price given by the put contract. The value of the put when exercised is equal to the amount by which the put exercise price exceeds the underlying stock price (or zero if the put is never exercised).

To continue the above example, suppose that there is a put option with an exercise price of $90 on one share of stock. The put option expires in one year. Again, this share of stock is expected to be worth either $80 or $120 in one year, but we do not know which yet. If the stock were to be worth $80 when the put expires, its owner should exercise the put. In this case, its owner could use the put to sell stock for $90 (the exercise price) when it can be purchased in the market for $80. The put would be worth $10 in this case. If, instead, the stock were to be worth $120 when the put expires, its owner should not exercise the put. Its owner should not accept $90 for a share that has a market value of $120. In this case, the call would be worth nothing when it expires. Let p_T be the value of the put option at expiry, determined as follows:

(2) $$p_T = MAX[0, X - S_T]$$

When S_T=80, p_T = MAX[0, 90 – 80] = 10
When S_T=120, p_T = MAX[0, 90 – 120] = 0

Thus, Table A.1 can be rewritten for our example as Table A.2. In Table A.2, the total Time 1 payoff from purchasing and selling the underlying asset is either 80 or 120. The total Time 1 payoff from short selling and then repurchasing is either -80 or -120. The short seller sells to the buyer at Time 0; the buyer sells to the short seller at Time 1. The short-seller must repurchase the stock.

The Time 1 profit from purchasing the call, ignoring the Time 0 premium paid at purchase, is either 0 = MAX[0, 80-90] or 30 = MAX[0,120-90]; in the first instance, the call is disposed of, in the second, the call is exercised. The Time 1 profit from selling (writing) the call, ignoring the Time 0 premium, is either 0 = -MAX[0, 80-90] or -30 = -MAX[0,120-90].

Your Position	Payoff if $S_1 \leq 90$	Payoff if $S_1 > 90$	Notes on your Position
Long Underlying	80	120	You own the underlying asset
Short Underlying	- 80	- 120	You short sold the underlying asset
Long Call	0	30	You dispose of or exercise the call
Short Call	0	- 30	You are obliged to allow exercise
Long Put	10	0	You exercise or dispose of the put
Short Put	- 10	0	You are obliged to allow exercise

Table A.2: Stock and Plain Vanilla Option Position Payoffs Example

The Time 1 profit from purchasing the put, ignoring the Time 0 premium at its sale, is either 10 = MAX[0, 90-80] or 0 = MAX[0,90-120]; in the first instance, the put is exercised, in the second, the put is disposed of. The Time 1 profit from selling (writing) the put, ignoring the Time 0 premium, is either -10 = -MAX[0, 90-80] or 0 = -MAX[0,90-120].

Long for Option; Short for Obligation

The owner of the option contract may exercise his right to buy or sell; however, he is not obligated to do so. Stock options are simply contracts between two investors issued with the aid of a clearing corporation, exchange and broker, which ensure that investors honor their obligations to each other. The corporation whose stock options are traded will probably not issue and does not necessarily trade these options. Investors, typically through a clearing corporation, exchange and brokerage firm, create and trade option contracts amongst themselves.

For each owner of an option contract, there is a seller or "writer" who creates the contract, sells it to a buyer and must satisfy an obligation to the owner of the option contract. The option writer sells (in the case of a call exercise) or buys (in the case of a put exercise) the stock when the option owner exercises. The owner of a call is likely to profit if the stock underlying the option increases in value sufficiently over the exercise price of the option (he can buy the stock for less than its market value); the owner of a put is likely to profit if the underlying stock declines in value sufficiently below the exercise price (he can sell stock for more than its market value). Since the option owner's right to exercise represents an obligation to the option writer, the option owner's profits are equal to the option writer's losses. Therefore, an option must be purchased from the option writer; the option writer receives a "premium" from the option purchaser for assuming the risk of loss associated with enabling the option owner to exercise. Next, we begin the process of determining the call and put values at time zero.

2. Derivative Securities Markets and Hedging

As discussed earlier, a *derivative security* is simply a financial instrument whose value is derived from that of another security, financial index, or rate. A large number of different types of derivative securities have become increasingly important for management and hedging a variety of different types of risks. There exist a huge variety of derivative securities. Some of the more frequently traded derivatives follow:

Futures contracts, which provide for the transfer of a given asset at an agreed-to price at a future date.
Options, which confer the right but not obligation to buy or sell an asset at a prespecified price on or before a given date.
Swaps, which provide for the exchange of one set of cash flows for another set of cash flows.
Hybrids, which combine features of two or more securities (e.g., a convertible bond).

Risk factors frequently hedged with derivatives include, but are not limited to, uncertainties associated with underlying and related asset price movements, interest and exchange rate variability, debtor default, and economy-wide and sector-specific output levels. Markets for explicit insurance policies on such a wide array of risks do not exist largely due to contracting costs. Most insurance policies are fairly standardized (e.g., health, life, and many casualty policies), while customized insurance contracts are expensive and time consuming to write. Businesses must be able to act quickly to manage their risks in this environment of rapid change. Flexibility and liquidity along with low contracting and transactions costs are key to the success of the risk management operations of a firm. An active and efficient market for derivative securities can help firms hedge and manage a variety of different types of risks.

Business firms and individual investors desiring to hedge risks are not the only participants in markets for derivatives. A second type of market participant is the speculator who takes a position in a security based on his expectation regarding future price movement. Although the speculator is essentially concerned with his own trading profits, he plays an important role in maintaining liquidity in derivative markets, affording business and individual investors the opportunity to hedge risks quickly and efficiently. The speculator is often the counterparty to a hedger's trade, selling or purchasing derivatives as required by hedgers.

The arbitrageur, who exploits situations where derivatives are mispriced relative to one another, not only provides additional liquidity to derivative markets, but plays an important role in their pricing. By constantly seeking price misalignments for a variety of types of securities, and by understanding the payoffs of securities relative to one another, arbitragers help ensure that derivative securities are fairly priced. This activity reduces price volatility and uncertainty faced by hedgers. In addition, the presence of arbitrageurs provides liquidity to other market participants.

As discussed earlier, derivative securities are traded in the United States either on exchanges or over the counter (OTC) markets. Substantial market interest is required for exchange listing, whereas securities with smaller followings or even customized contracts can be traded OTC, including trading between banks and other major participants. The role of the derivatives dealer is essentially the same as that for other security dealers: to facilitate transactions for clients at competitive prices. Derivative dealers match counterparties for derivative contracts, act as counterparty for many of their own custom contracts, and provide an array of support services including expert advice and carefully engineered customized risk management products. It is necessary that the dealer providing full support services have a proper understanding of his client and the client's business and needs. Since some clients do not have an understanding of the technical terms used in the industry, the dealer must be an effective communicator. It is equally important for the dealer to understand the nature of the securities with which he deals and how serving as a market maker for derivatives affects the risk structure of his employer. This understanding usually requires strong analytical skills.

Many stock options in the United States and Europe are traded on exchanges. The largest U.S. stock options exchanges are the Chicago Board Options Exchange and the NASDAQ OMX Group. The NYSE operates the options markets of the former American Exchange and the International Stock Exchange maintains an options market. Options are also traded on numerous commodities, futures contracts, currencies, and other financial instruments such as Treasury instruments and index contracts.

Options can be classified into either the European variety or the American variety. European options may be exercised only at the time of their expiration; American options may be exercised any time before and including the date of expiration. Most option contracts traded in the United States (and Europe as well) are of the American variety. American options can never be worth less than their otherwise identical European counterparts. In fact, because most call options have time value in addition to their intrinsic or exercise value (calls on stocks that go ex-dividend before the call expires can be an important exception to this), we usually do not exercise American calls before exercise. This means that we can often value American call options (on non-dividend-paying stock) as though they are European calls.

3. Put-Call Parity

First, since the call (put) owner has the right to buy (sell) the underlying stock at a price of X, the terminal payoff functions for calls and puts are written as follows:

$$c_T = MAX[0, S_T - X]$$
$$p_T = MAX[0, X - S_T]$$

If we subtract the second equation 2 above from the first, we obtain the terminal value *put-call relation*:

$$c_T - p_T = MAX[0, S_T - X] - MAX[0, X - S_T] = S_T - X$$

A slight rewrite of this terminal put-call relation allows us to write the terminal or exercise value a put given the terminal value of a call with identical exercise terms:

$$p_T = c_T + X - S_T$$

Since the terminal value of a put is always given by this equation, the time zero value of a put must be given by:

$$p_0 = c_0 + X \times e^{-r_f T} - S_0$$

In continuous time or $p_0 = c_0 + X \times \frac{e}{1+r_f}^{T} - S_0$ in discrete time. This arbitrage-free relationship allows us to value a put based on the price knowledge of a call with exactly the same exercise terms. The put-call parity function holds regardless of the stochastic process generating stock prices, but assumes that the underlying stock pays no dividends during the lives of the options.

Illustration: Hedging With a Call, a Put and a Collar

Suppose that an investor has 1000 shares of stock that are currently worth \$50 per share each. The investor wishes to lock in his gains on the stock but does not want to sell at this time for tax reasons. The current two-year riskless return rate is 5%. The investor could write two-year calls on these shares, with an exercise price of \$50 per share, receiving \$8 for each share that he owns. That is, he could receive \$8 per share by selling covered calls. He could also purchase two-year puts on each share which would enable him to eliminate downside risk. What are the potential hedging strategies available to the investor? What are the costs and benefits of each?

First, the investor could simply write covered calls. This strategy offers no downside risk protection, but it does provide the investor with \$8 (or \$8,000 total) in immediate cash flows. The strategy also limits the investor's upside potential on the shares, locking out any gains should the share price rise above \$50. This is because the investor would be obliged to sell the stock to the owner of the call should its price rise above \$50 per share. The owner of the call would not exercise this option if the share price remains below \$50. While the covered call strategy does reduce portfolio volatility, it is more significant to the investor as a means of generating short-run income at the expense of potential stock profits.

A second potential strategy for the investor is to purchase puts. We can use put-call parity to value these puts. Since the riskless rate is $r_f = 0.05$, the exercise price is X=\$50, the current share price is S_0=\$50, and the T= 2 and the two-year call price is c_0=\$8, we can use put-call parity to value the put:

$$p_0 = 8 \;+\; 50 \times e^{-.05 \times 2} - 50 \;=\; 3.24$$

Each put will protect the investor from downside stock price movements below \$50 per share, and, based on the data given, a fair price for each put is \$3.24. Thus, for \$3.24, the investor can lock in a minimum selling price of \$50 for each share. The puts will not require the investor to give up any gains above \$50.

What if the investor objects to paying \$3.24 to eliminate downside price risk? Another possibility is to simultaneously purchase a put and write a call, in effect financing the cost of the put with proceeds from selling the call. One such strategy is to create a collar, which is a package consisting of a long position in a put and a short position in a call. Here, if the investor purchases a put for \$3.24 and sells a call for \$8, he nets \$4.76 per share. If the share price drops below \$50 in two years, he puts his shares to the put writer for \$50. If the share price rises above \$50 in two years, his shares are called away from him at \$50 per share. Thus, for a net cash flow today of \$4.76, the investor gives up all potential profits and losses on his shares, locking or "collaring" in a price of \$50 today.

4. Options and Hedging in a Binomial Environment

The Binomial Option Pricing Model is based on the assumption that the underlying stock price is a Bernoulli trial in each period, such that it follows a binomial multiplicative return generating process. This means that for any period following a particular outcome, the stock's value will be only one of two potential constant values. For example, the stock's value at time t+1 will be either u (multiplicative upward movement) times its prior value S_t or d (multiplicative downward movement) times its prior value S_t.

Notice that we have not specified probabilities of a stock price increase or decrease during the period prior to option expiration. Nor have we specified a discount rate for the option or made inferences regarding investor risk preferences. We will value this call based on the fact that during this single time period, we can construct a riskless hedge portfolio consisting of a position in a single call and offsetting positions in α shares of stock. This means that by purchasing a single call and by selling α shares of stock, we can create a portfolio whose value is the same regardless of whether the underlying stock price increases or decreases. The ratio of the number of shares to offset each call in the portfolio is called the *hedge ratio*, or in the multi-period framework, the *dynamic hedge*. Let us first define the following terms:

X = Exercise price of the stock
S_0 = Initial stock value
u = Multiplicative upward stock price movement
d = Multiplicative downward stock price movement
c_u = MAX[0,uS_0 - X]; Value of call if stock price increases
c_d = MAX[0,dS_0 - X]; Value of call if stock price decreases
α = Hedge ratio
r = Riskless return rate

<u>The Hedge Ratio</u>

In a one-time period binomial framework where there exists a riskless asset, we can hedge the call against the stock such the resultant portfolio with one call and α shares of stock produces the same cash flow whether the stock increases or decreases:

(1) $$c_u + \alpha u S_0 = c_d + \alpha d S_0 = (c_0 + \alpha S_0)(1+r)$$

This one-time period hedge implies a hedge ratio α, which provides for the number of shares per call option position to maintain the perfect hedge (portfolio that replicates the bond):

(2) $$\alpha = \frac{c_u - c_d}{S_0(d-u)}$$

Negative values for α, the hedge ratio, which will always be the case when the call is purchased, imply that α shares of stock are shorted for each call that is purchased.

Multi-period models lead to 2^T potential outcomes without recombining, or, in many instances, *T+1* potential outcomes with recombining. Thus, complete capital markets requires a set of 2^T or *T* priced securities (stocks or options) with payoff vectors in the same payoff space such that the set of payoff vectors is independent. In multiple period frameworks, hedging with many securities guarantees hedged portfolios at the portfolio termination or liquidation dates. In a binomial framework, there is an important exception. If the hedge α_t can be updated each period *t*, we write the hedge ratio for period *t* as follows:

$$\alpha_t = \frac{c_{u,t} - c_{d,t}}{S_t(d-u)}$$

The dynamic hedge is updated each period. Any portfolio employing and updating this dynamic hedge in each period in a binomial framework will be riskless at each period.

Pricing the Call in the One-Period Case

Since the risk of a call can be hedged with an offsetting position in a call, we can perform a little algebra on Equation (1) to value the call in Equation (3) as follows:

(1) $$c_d + \alpha d S_0 = (c_0 + \alpha S_0)(1+r)$$

(3) $$c_0 = \frac{-\alpha S_0(1+r) + c_d + \alpha d S_0}{(1+r)}$$

Equation (3) is the binomial model for one-period pricing for the call. We can add some insight to this by filling in the hedge ratio from Equation 2 as follows, then rearrange terms to simplify and to separate c_u and c_d as follows in Equation (4):

$$c_0 = \frac{-\frac{c_u - c_d}{S_0(d-u)} S_0(1+r) + c_d + \frac{c_u - c_d}{S_0(d-u)} dS_0}{(1+r)}$$

$$c_0 = \frac{\frac{c_u}{(u-d)}(1+r) - \frac{c_d}{(u-d)}(1+r) + \frac{c_d(u-d)}{(u-d)} - \frac{c_u}{(u-d)} d + \frac{c_d}{(u-d)} d}{(1+r)}$$

(4) $$c_0 = \frac{c_u\left[\frac{(1+r)-d}{(u-d)}\right] + c_d\left[\frac{u-(1+r)}{(u-d)}\right]}{(1+r)}$$

Equation (4) has a very nice feature: c_0 can be expressed as the discounted expected present value of the call, which equals the upjump value of the call multiplied by some function of *r*, *u* and *d* plus the downjump value of the call multiplied by some other function of *r*, *u* and *d*. These two functions can be interpreted as risk-neutral probabilities, which is what we will do in the next subsection.

Risk-Neutral Setting: One-Period Case

Here, we will calculate the risk-neutral probabilities. In earlier chapters, we valued a call as payoff structure identical to a portfolio comprised of underlying shares and bonds. This is what we did above in Equation (3) with the hedge ratio. We use a similar method here, where the equivalent martingale measure is calculated from bond and stock payoffs, and the call is valued based on those risk-neutral probabilities. Valuing the one-time period call in order to obtain its risk-neutral pricing in a single time period binomial framework is straightforward given the risk-neutral probability measure $\mathbb{Q}$, where *q* represents the risk-neutral probability (from the equivalent martingale $\mathbb{Q}$) of an upjump:

$$E_{\mathbb{Q}}[c_1] = [c_u q + c_d(1-q)] = c_0(1+r).$$

Solving for c_0 gives the price:

$$c_0 = \frac{c_u q + c_d(1-q)}{1+r}$$

We value shares of the stock relative to the bond as follows:

$$E_{\mathbb{Q}}[S_1] = uS_0 q + dS_0(1-q) = S_0(1+r),$$

which implies that:

$$q = \frac{1+r-d}{u-d}$$

Part of what makes this model so useful is that we do not need to know investor risk-return preferences, the expected return for the stock or even the physical probabilities for upjumps and downjumps. This last point is worth emphasizing: We do not need to know physical probabilities. Instead, we can calculate risk neutral probabilities for upjumps q and downjumps (1-q) based on the equivalent martingale. Now, we see that we can price the call in a one-period environment with either of the following:

$$c_0 = \frac{c_u q + c_d(1-q)}{1+r}$$

<u>Illustration: Binomial Option Pricing - One Period Case</u>

Consider a stock currently selling for 10 and assume for this stock that u equals 1.5 and d equals .5 (This is a continuation of our numerical example from Section 5.2.2 above). The stock's value in the forthcoming period will be either 15 (if outcome u is realized) or 5 (if outcome d is realized). Consider a one-period European call trading on this particular stock with an exercise price of 9. If the stock price were to increase to 15, the call would be worth 6 ($c_u = 6$); if the stock price were to decrease to 5, the value of the call would be zero ($c_d = 0$).In addition, recall that the current riskless one-year return rate is .25. Based on this information, we should be able to determine the value of the call.

In our numerical example offered above, we use the following to determine the value of the call in the binomial framework:

$S_0 = 10$	$u = 1.5$	$d = .5$
$c_u = 6$	$c_d = 0$	$r = .25$
$X = 9$		

The risk-neutral probability of an upjump, the hedge ratio and the time zero value of the call in the one-period framework are calculated as follows:

$$q = \frac{1+r-d}{u-d} = \frac{1+.25-.5}{1.5-.5} = .75$$

$$\alpha = \frac{c_u - c_d}{S_0(d-u)} = \frac{6-0}{10(.5-1.5)} = -.6$$

$$c_0 = \frac{c_u q + c_d(1-q)}{1+r} = \frac{6\times.75 + 0\times(1-.75)}{1+.25} = 3.6$$

Recall that the time zero value of the bond was .8. The time zero value of the call is 3.6/.8 = 4.5 times that of the bond; the time zero expected value of the call at time 1 is also 4.5 times the value of the time one bond value:

$$E_{\mathbb{Q}}[c_1] = [c_u q + c_d(1-q)] = 6\times.75 + 0\times(1-.75) = 4.5$$

Thus, under probability measure $\mathbb{Q}$ with the bond as the numeraire, the call price process is a martingale, just as the stock price process is.

B. Multi-Period Framework

Suppose that we express our outcomes in terms of *u* and *d*, such that the numbers of upjumps and downjumps over time determine stock prices. If we are willing to assume that the probability measure is the same in each of these *T* time periods, by invoking the Binomial Theorem, we see that valuing the call in the multi-period binomial setting is similarly straightforward:[1]

$$c_0 = \frac{\sum_{j=0}^{T}\frac{T!}{j!(T-j)!}q^{\,j}(1-q)^{T-j}MAX[u^j d^{T-j}S_0 - X, 0]}{(1+r)^T}$$

The number of computational steps required to solve this equation is reduced if we eliminate from consideration all of those outcomes where the option's expiration date price is zero. Thus, *a*, the smallest non-negative integer for j where $S_T > X$ is given as follows:[2]

(19)
$$a = INT\left[MAX\left[\frac{\ln\left(\frac{X}{S_0 d^T}\right)}{\ln\left(\frac{u}{d}\right)}, 0\right] + 1\right]$$

We can simplify the Binomial Model further by substituting *a* and rewriting as follows:

(20)
$$c_0 = \frac{\sum_{j=a}^{T}\frac{T!}{j!(T-j)!}q^j(1-q)^{T-j}\left[u^j d^{T-j}S_o - X\right]}{(1+r)^T}$$

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[1] See end-of chapter exercise 3 for a derivation. See also Cox and Rubenstein [1985].

[2] We obtain *a* by first determining the minimum number of price increases *j* needed for S_T to exceed *X*:

$$S_T = u^j d^{T-j} S_0 > X$$

We then solve this inequality for the minimum positive integer value for *j* with $u^j d^{T-j} S_0 > X$ (note: if $j \leq 0$, *a* will equal 0). First divide both sides of the inequality by $S_0 d^T$ so that $(u/d)^j > X/(S_0 d^T)$. Next, take logs of both sides to obtain: $j\ln(u/d) > \ln(X/(S_0 d^T))$. Finally, divide both sides by $ln(u/d)$ to get the desired result. Thus, *a* is the smallest positive integer for *j* such that $S_T > X$.

or:

$$c_0 = S_0\left[\sum_{j=a}^{T}\frac{T!}{j!(T-j)!}q^j(1-q)^{T-j}\frac{u^j d^{T-j}}{(1+r)^T}\right]-\frac{X}{(1+r)^T}\left[\sum_{j=a}^{T}\frac{T!}{j!(T-j)!}q^j(1-q)^{T-j}\right]$$

or, in short-hand form:[3]

$$c_0 = S_0 B[T,q'] - X(1+r)^{-T}B[T,q]$$

where $q' = qu/(1+r)$ and $1-q' = d(1-q)/(1+r)$. The values q', q and T are the parameters for the two binomial distributions. Three points are worth further discussion regarding this simplified Binomial model:

1. First, as T approaches infinity, the binomial distribution will approach the normal distribution, and the binomial model will approach the Black Scholes model, which we will discuss later in this chapter and in Chapter 6.
2. The current value of the option is:

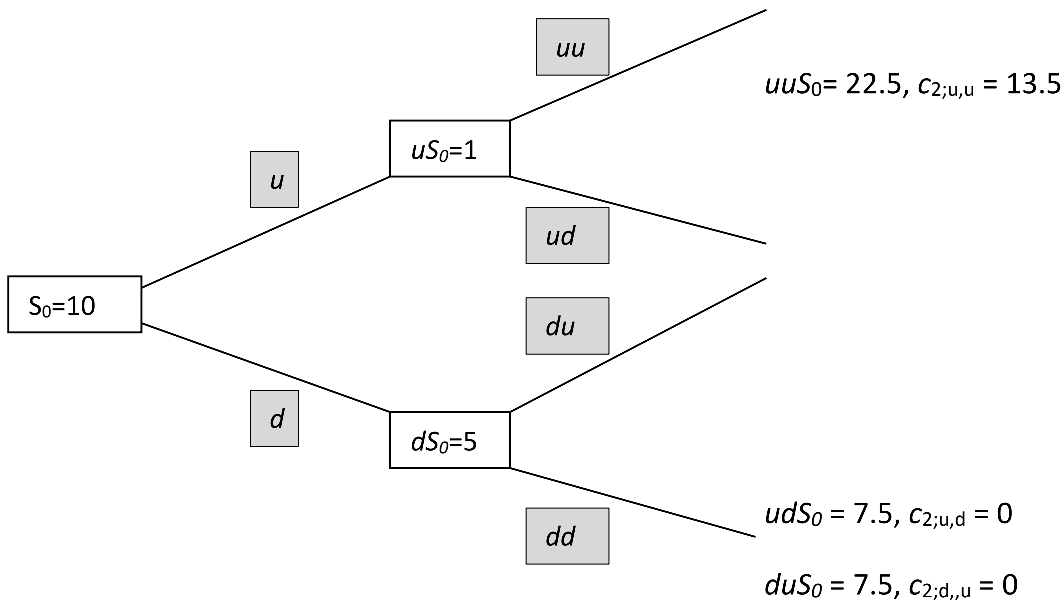
$$c_0 = \frac{E[c_T]}{(1+r)^T} = P[S_T > X]\frac{E[S_T|S_T > X]}{(1+r)^T} - \frac{X}{(1+r)^T}P[S_T > X]$$

   First, this implies that the binomial distribution $B[T,q] = P[S_T > X]$ provides the probability that the stock price will be sufficiently high at the expiration date of the option to warrant its exercise. Second, $S_0B[T, q']/B[T, q]$ can be interpreted as the discounted expected future value of the stock conditional on its value exceeding X.
3. The call is replicated by a portfolio comprised of a long position in $B[T,q'] < 1$ shares of stock and borrowings. Investment in stock totals $S_0B[T,q']$ and borrowings total $X(1+r)^{-T}B[T, q]$. The replication amounts must be updated at each time period.

Extending the Binomial Model to Two Periods

Now, we will extend our illustration above from a single period to two periods, each with a riskless return rate equal to .25. As before, the stock currently sells for 10 and will change to either 15 or 5 in one time period (u = 1.5, d = .5). However, in the second period, the stock will change a second time by a factor of either 1.5 or .5, leading to potential values of either 22.5 (up then up again), 10 (up once and down once) or 2.5 (down twice).Recall the lattice associated with this stock price process is depicted in Figure 1 above. These stock prices are listed with call option values in Figure 2.

[3]As the lengths of time periods approach zero, d must approach $1/u$.



Since *u*, *d* and *r* are the same for each period, probability measure $\mathbb{Q}$ will be the same for each period. Thus, $q = .75$ and $(1\text{-}q)= .25$ in each each period. However, the hedge ratio α_t will change for each period, depending on the share price movement in the prior period:

$$\alpha_0 = \frac{c_{1;u} - c_{1;d}}{S_0(d-u)} = \frac{6-0}{10(.5-1.5)} = -.6$$

$$\alpha_{1,u} = \frac{c_{2;u,u} - c_{2;u,d}}{uS_0(d-u)} = \frac{13.5-0}{15(.5-1.5)} = -.9$$

$$\alpha_{1,d} = \frac{c_{2;d,u} - c_{2,d,d}}{dS_0(d-u)} = \frac{0-0}{7.5(.5-1.5)} = 0 \quad (no\ hedge)$$

Thus, the hedge ratio must be adjusted after each price change. Actually, there is no hedge or hedge ratio after the stock price decreases after time zero. The call option has no value in this event and cannot be used to hedge the stock risk. The hedge ratio for Time 1, assuming that the stock price increased after time zero will be -.9, meaning that .9 shares of stock must be short sold for each purchased call to maintain the hedged portfolio.

The time-zero 2-period binomial call price is calculated as follows:

$$c_0 = \frac{\sum_{j=0}^{2}\frac{2!}{j!(2-j)!}.75^{\,j}(1-.75)^{2-j}MAX\left[1.5^j.75^{2-j}\times 10-9,0\right]}{(1+.25)^2} = \frac{c_{2;uu}\,q^{\,2}}{(1+r)^2} = \frac{13.5\times.5625}{1.5625} = 4.86$$

Notice that, in the two-period framework, the call has the same value at time zero (4.86) as 7.59375 bonds at time zero. In addition, its time zero expected value in time 2 is also the same as 7.59375 bonds:

$$E_{\mathbb{Q}}[c_2] = [c_{2;u,u}\,q^{\,2} + 2c_{2;u,d}q\,(1-q\,) + c_{2;d,d}\,(1-q\,)^2] = (22.5-9)\times.5625 = 7.59375$$

Thus, under the 2-period probability measure $\mathbb{Q}$ with the bond as the numeraire, the call price process is a martingale, just as the stock price process is:

$$E_{\mathbb{Q}}[S_{0,2}] = [u^2S_0q_u{}^2 + 2udS_0q\,(1-q\,) + d^2S_0(1-q\,)^2] = 15.625 = S_0/b_0,$$

where the current value of the stock in this two-time period framework is 15.625 times the time zero value (.64) of the bond.

<u>Extending the Model to Three Periods</u>

Now, we will extend our illustration above from two periods to three, each with a riskless return rate equal to .25. Two upjumps are necessary for the call to be exercised. By the third period, potential stock values are either $1.5^3 \times 10 = 33.75$, $1.5^2 \times .5 \times 10 = 11.25$, $.5^2 \times 1.5 \times 10 = 3.75$ or $.5^3 \times 10 = 1.25$. The time zero call value in this three-period binomial framework is computed with Equations 19 and 18 as follows:

$$a = MAX\left[INT\left(\frac{ln\left[\frac{9}{10\times.5^3}\right]}{ln\left[\frac{1.5}{.5}\right]}+1\right),0\right]=2$$

$$c_0 = \frac{[3\times.75^2\times.25\times(1.5^2\times.5\times10-9)]+[1\times.75^3\times(1.5^3\times10-9)]}{(1+.25)^3} = 5.832$$

Put-call parity still applies:[4]

$$p_0 = 5.832 + 9/(1.25)^3 - 10 = 0.44$$

C. Multiplicative Upward and Downward Movements in Practice

[4] We will discount the exercise money with a discrete discount function since the binomial model is a discrete time model.

One apparent difficulty in applying the binomial model as it is presented above is in obtaining estimates for u and d that are required for p; all other inputs are normally quite easily obtained. There are several methods that are used to obtain parameters for the binomial method from the actual security returns generating process. For sake of simplicity here, we will assume that all investors are risk-neutral, and that physical probabilities and their martingale equivalents are the same ($p = q$). For example, following Cox, Ross and Rubinstein [1979] derive the following to estimate probabilities of an uptick p and downtick (1 - p):[5]

(20) $$p = \frac{e^{r_f} - d}{u - d} \qquad (1-p) = \frac{u - e^{r_f}}{u - d}$$

Cox *et al.* also proposes the following to estimate u and d in the Binomial approximation to the Wiener process, where σ is the standard deviation of stock returns:

(21) $$u = e^{\sigma} \qquad d = \frac{1}{u}$$

or, if n and T differ from 1:

(22) $$u = e^{\sigma\sqrt{\frac{T}{n}}} \qquad d = \frac{1}{u}$$

Suppose, for example, that for a particular Wiener process, $\sigma = .30$ and $r_f = .05$. Using Equations 20 and 22 above, we estimate p, u and d for a single time period binomial process as follows:

$$u = e^{.3} = 1.3498588$$

$$d = \frac{1}{u} = .7408182$$

The Binomial Model in Practice: An Illustration

Suppose that we wished to evaluate a call and a put with an exercise price equal to \$110 on a share of stock currently selling for \$100. The underlying stock return standard deviation equals .30 and the current riskless return rate equals .05. If both options are of the European variety and expire in six months, what are their values?

First, we will compute the call's value using the binomial model. We will vary the number of jumps in the model as the example progresses. First, let $n = 1$ and use Equations 20 and 22 to compute p, u and d:

$$u = e^{.3\times\sqrt{.5}} = 1.2363111$$

[5] See Appendix A for derivations of p, u and d. Regardless, notice the somewhat minor deviation from the probability estimates given by Equation set 14. The difference between Equations sets 14 and 20 is that Equation set 20 allows interest (r_f) to be continuously compounded whereas Equation set 14 is based on discrete compounding. This distinction is not important for our purposes here.

$$d = \frac{1}{1.2363111} = .8088578$$

$$p = \frac{e^{r_f T} - d}{u - d} = 0.5063881$$

Thus, in a risk-neutral environment, there is a .5064 probability that the stock price will increase to 123.63 and a .4936 probability that the stock price will be 80.88678. Similarly, in a risk neutral environment, there is a .5064 probability that the call will be worth 13.63; therefore, its current value is 6.73 = .5064 × 13.63 × $e^{-.05\times.5}$. The call value is determined by the binomial model as follows where $n = a = 1$:[6]

$$c_0 = \frac{\sum_{j=a}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \left[u^j \times d^{n-j} S_o - X\right]}{\left(1+r_f\right)^T}$$

$$c_0 = \frac{.506388^1 \times .4936119^{1-1} \times \left[1.2363111^1 \times .8088578^{1-1} \times 100 - 110\right]}{\left(1+.05\right)^{.5}} = 6.73$$

We can also use the binomial model to value the put with identical exercise terms on the underlying stock:[7]

$$p_0 = \frac{\sum_{j=0}^{a-1} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \left[X - u^j d^{n-j} S_o\right]}{\left(1+r_f\right)^T}$$

$$p_0 = \frac{.506388^0 \times .4936119^1 \times \left[110 - 1.23631^0 \times .8088578^1 \times 100\right]}{\left(1+.05\right)^{.5}} = 14.08$$

---

[6] a is determined by Equation 16 and is the first positive integer where $u^j d^{n-j} S_0 > X$. That is, the minimum number of up-jumps required for exercise of the call option equals a. Any smaller number of stock up-jumps produces a terminal call value equal to zero, and need not be considered.

[7] Recall that put-call parity, demonstrated in Section C, can be used to value puts. We verify this for this example by using Equation 3 as follows: 14.08 = 6.73 + 110$e^{-.05\times.5}$ - 100.

Dividing an Interval into Sub-Intervals

Now, divide the single six-month interval into two three-month intervals; that is, $n = 2$. We will now use a two-period binomial model to evaluate calls and puts on this stock. First, we use Equations 20 and 22 to compute p, u and d:[8]

$$u = e^{..3\sqrt{\frac{.5}{2}}} = 1.1618342$$

$$d = \frac{1}{1.1618342} = .8607079$$

$$p = \frac{e^{r_f T/n} - d}{u - d} = .5043415$$

Thus, there is a $.5043^2$ probability that the stock price will increase to 134.98, a .5 probability that the stock price will remain unchanged at 100 and a $.4957^2$ probability that the stock price will decline to 74.08. Thus, there is a .2543 probability that the call will be exercised, in which case, it will be worth 24.98. Therefore, the call's current value is $6.20 = .2543 \times .24.98 \times e^{-.05 \times .5}$. Call and put values are determined by the binomial model as follows where $n = a = 2$:

$$c_0 = \frac{\sum_{j=a}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \left[u^j d^{n-j} S_o - X\right]}{\left(1+r_f\right)^T}$$

$$c_0 = \frac{.5043^2 \times .4957^{2-2} \times \left[1.16185^2 \times .8607^{2-2} \times 100 - 110\right]}{\left(1+.05\right)^{.5}} = 6.20$$

[8] When there are multiple jumps per period (n > 1), and/or when T does not equal one, $p = \frac{e^{r_f (T/n)} - d}{u - d}$ and $\left(1-p\right) = \frac{u - e^{r_f (T/n)}}{u - d}$.

$$p_0 = \frac{\sum_{j=1}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \left[X - u^j d^{n-j} S_o\right]}{\left(1+r_f\right)^T}$$

$$p_0 = \frac{.5043^{2-2} \times .4957^2 \times \left[110 - 1.1618^{2-2} \times .8607^2 \times 100\right] + 2 \times .5043 \times .4957 \times [110-100]}{(1+.05)^{.5}}$$

$$= 6.20 + 110e^{-.05\times.5} - 100 = 13.48$$

As the six-month period is divided into more and finer subintervals, the values of the call and put will approach their Black-Scholes values. Table 1 extends this example to more than two subintervals, ultimately approaching the Black-Scholes model.

n	c_0	p_0
1	6.73	14.02
2	6.20	13.48
3	5.47	12.72
4	6.04	13.30
5	5.18	12.44
6	5.91	13.17
7	5.43	12.68
8	5.81	13.06
9	5.57	12.82
10	5.73	12.98
50	5.63	12.89
100	5.59	12.85
∞	5.59	12.85

Volatility	$\sigma = .30$
Riskless rate	$r_f = .05$
Exercise price	$X = 110$
Initial stock price	$S_0 = 100$
Term to expiration	$t = .5$

Table 1: Convergence of the Binomial Model to the Black-Scholes Model

5. A Primer on Black-Scholes Pricing

The Black-Scholes Options Pricing Model provides a simple mechanism for valuing calls under certain assumptions. If circumstances are appropriate to apply the Black-Scholes model, call options can be valued with the following:

(3)

$$c_0 = S_0 N(d_1) - \frac{X}{e^{r_f T}} N(d_2)$$

(4)

$$d_1 = \frac{ln\left(\frac{S_0}{X}\right) + \left(r_f + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

(5)

$$d_2 = d_1 - \sigma\sqrt{T}$$

where N(d*) is the cumulative normal distribution function for (d*). This function might be referred to in a statistics setting as the "z" value for (d*). From a computational perspective, one would first work through Equation (4), then Equation (5) before valuing the call with Equation (3). $N(d_1)$ and $N(d_2)$ are areas under the standard normal distribution curves (z-values). Simply locate the z-value on an appropriate table (see the z-table in chapter Appendix B) corresponding to the $N(d_1)$ and $N(d_2)$ values.

Consider the following simple illustration of a Black-Scholes Model application: An investor has the opportunity to purchase a six month call option for $7.00 on a stock which is currently selling for $75. The exercise price of the call is $80 and the current riskless rate of return is 10% per annum. The variance of annual returns on the underlying stock is 16%. At its current price of $7.00, does this option represent a good investment? First, we note the model inputs in symbolic form:

$T = .5$ $r_f = .10$ $\sigma = .4$ $S_0 = 75$
$X = 80$ $\sigma^2 = .16$

Our first steps are to find d_1 and d_2 from Equations 4 and 5:

$$d_1 = \frac{ln\left(\frac{75}{80}\right) + \left(0.10 + \frac{1}{2}.4^2\right) \times 0.5}{0.4 \times \sqrt{.5}} = \frac{ln(0.9375) + 0.09}{0.2828} = 0.09$$

$$d_2 = 0.09 - 0.4 \times \sqrt{.5} = 0.0909 - .2828 = -0.1928$$

Next, by either using a z-table (see Table A.4.a in the text Appendix) or by using an appropriate estimation function from a statistics manual, we find normal density functions for d_1 and d_2:

$$N(d_1) = N(0.09) = 0.536; \quad N(d_2) = N(-0.1928) = 0.424$$

Finally, we use $N(d_1)$ and $N(d_1)$ in Equation (3) to value the call:

$$c_0 = 75 \times 0.536 - \frac{80}{e^{.10 \times .5}} \times 0.42 = 7.96$$

Since the 7.96 estimated value of the call exceeds its 7.00 market price, the call should be purchased.

Put-Call Parity

Appendix Exercises

1. Call and put options with an exercise price of $30 are traded on one share of Company X stock.
   a. What is the value of the call and the put if the stock is worth $33 when the options expire?
   b. What is the value of the call and the put if the stock is worth $22 when the options expire?
   c. What is the value of the call writer's obligation stock is worth $33 when the options expire? What is the value of the put writer's obligation stock is worth $33 when the options expire?
   d. What is the value of the call writer's obligation stock is worth $22 when the options expire? What is the value of the put writer's obligation stock is worth $22 when the options expire?
   e. Suppose that the purchaser of a call in part a paid $1.75 for his option. What was the profit on his investment?
   f. Suppose that the purchaser of a call in part b paid $1.75 for his option. What was the profit on his investment?

2. An investor has the opportunity to purchase a three-year call option on a stock that is currently selling for $150. The exercise price of the call is $140 and the current riskless rate of return is 2% per annum. The variance of annual returns on the underlying stock is 16%. What is the value of this call?

3. Evaluate calls for each of the following European stock option series:

Option 1	Option 2	Option 3	Option 4
$T = 1$	$T = 1$	$T = 1$	$T = 2$
$S = 30$	$S = 30$	$S = 30$	$S = 30$
$\sigma = .3$	$\sigma = .3$	$\sigma = .5$	$\sigma = .3$
$r_f = .06$	$r_f = .06$	$r_f = .06$	$r_f = .06$
$X = 25$	$X = 35$	$X = 35$	$X = 35$

Appendix Exercise Solutions

1. a. $c_T = \$33 - \$30 = \$3$; $p_T = 0$
   b. $c_T = 0$; $p_T = \$30 - \$22 = \$8$
   c. $c_T = -\$3$; $p_T = 0$
   d. $c_T = 0$; $p_T = -\$8$

e. \$3 - \$1.75 = \$1.25
f. \$0 - \$1.75 = -\$1.75

2. First, we note the model inputs in symbolic form:

$T = 3$ $r_f = .02$ $\sigma = .4$ $S_0 = 150$ $X = 140$ $\sigma^2 = .16$

Our first steps are to find d_1 and d_2:

$$d_1 = \frac{ln\left(\frac{150}{140}\right) + \left(0.02 + \frac{1}{2} \times .16\right) \times 3}{0.4 \times \sqrt{3}} = \frac{ln(1.07) + 0.3}{0{,}4 \times 1.736} = 0.5326$$

$$d_2 = 0.5326 - 0.4 \times \sqrt{3} = 0.5326 - .0.693 = -0.16$$

Next, by either using a z-table (see Table in the text Appendix) or by using an appropriate estimation function from a statistics manual or spreadsheet, we find normal density functions for d_1 and d_2:

$$N(d_1) = N(0.5326) = 0.7; \quad N(d_2) = N(-0.16) = 0.436$$

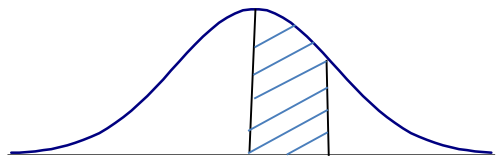
Finally, we use $N(d_1)$ and $N(d_1)$ to value the call:

$$c_0 = 150 \times 0.7 - \frac{140}{e^{.02 \times 3}} \times 0.436 = 48$$

3. The options are valued with the Black-Scholes Model in a step-by-step format in the following table:

	OPTION 1	OPTION 2	OPTION 3	OPTION 4
d(1)	.957739	-.163836	.061699	.131638
d(2)	.657739	-.463836	-.438301	-.292626
N[d(1)]	.830903	.434930	.524599	.552365
N[d(2)]	.744647	.321383	.330584	.384904
Call	7.395	2.455	4.841	4.623

Appendix B: z-table



The Normal Density Function
The z-Table

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	.0040	.0080	.0120	.0159	.0199	.0239	.0279	.0319	.0358
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0909	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2356	.2389	.2421	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2703	.2734	.2764	.2793	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3437	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3906	.3925	.3943	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4986	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

The areas given here are from the mean (zero) to z standard deviations to the right of the mean. To get the area to the left of z, simply add .5 to the value given on the table.

6. Estimating Implied Variances

Four of the five inputs required to implement the Black-Scholes model are easily observed. The option exercise price and term to expiry are defined by the option contract. The riskless return and underlying stock price are based on current quotes. Only the underlying stock return volatility during the life of the option cannot be observed. Instead, we often employ a traditional sample estimating procedure for return variance:

$$\sigma^2 = Var[r_t] = Var[lnS_t - lnS_{t-1}]$$

The difficulty with this procedure is that it requires that we assume that underlying security return variance is stable over time; more specifically, that future variances equal or can be estimated from historical variances. An alternative procedure first suggested by Latane and Rendleman [1976] is based on market prices of options that might be used to imply variance estimates. For example, the Black-Scholes Option Pricing Model might provide an excellent means to estimate underlying stock variances if the market prices of one or more relevant calls and puts are known. Essentially, this procedure determines market estimates for underlying stock variance based on known market prices for options on the underlying securities. When we use this procedure, we assume that the market reveals its estimate of volatility through the market prices of options.

Consider the following example pertaining to a six-month call currently trading for $8.20 and its underlying stock currently trading for $75:

$T = .5$	$r = .10$	$c_0 = 8.20$
$X = 80$	$S_0 = 75$	

If investors use the Black-Scholes Options Pricing Model to value calls, the following should be expected:

$$8.20 = 75N(d_1) - 80e^{-.1\times.5}N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{75}{80}\right) + \left(.1 + .5\times\sigma^2\right)\cdot .5}{\sigma\sqrt{.5}}$$

$$d_2 = d_1 - \sigma\sqrt{.5}$$

Through a process of substitution and iteration, we find that this system of equations holds when $\sigma = .41147$. Thus, the market prices this call as though it expects that the standard deviation of anticipated returns for the underlying stock is .41147.

Unfortunately, the system of equations required to obtain an implied variance has no closed form solution. That is, we will be unable to solve this equation set explicitly for standard deviation; we must search, iterate and substitute for a solution. One can substitute trial values for σ until she finds one that solves the system. A significant amount of time can be saved by using one of several well-known numerical search procedures such as the Method of Bisection or the Newton-Raphson Method.