CHAPTER FOUR

The Time Value of Money

4.1 Introduction and Future Value

The perspective and the organization of this chapter differs from that of chapters 2 and 3 in that topics are arranged by finance application rather than mathematics area. The mathematics tools presented in chapters 2 and 3 are applied in this chapter to closely examine the analytical aspects underlying what might be the single most important topic in finance – the *time value of money*. In this chapter, we study how investors and borrowers interact to value investments and determine interest rates on loans and fixed income securities.

Interest is paid by borrowers to lenders for the use of lenders' money. The level of interest charged is typically stated as a percentage of the *principal* (the amount of the loan). When a loan matures, the principal must be repaid along with any unpaid accumulated interest. In a free market economy, interest rates are determined jointly by the supply of and demand for money. Thus, lenders will usually attempt to impose as high an interest rate as possible on the money they lend; borrowers will attempt to obtain the use of money at the lowest interest rates available to them. Competition among borrowers and competition among lenders will tend to lead interest rates toward some competitive level. Factors affecting the levels of interest rates will do so by affecting supply and demand conditions for money. Among these factors are inflation rates, loan risks, investor intertemporal monetary preferences (how much individuals and institutions prefer to have money now rather than have to wait for it), government policies, and the administrative costs of extending credit.

4.2 Simple Interest

(Background reading: sections 2.4, 2.7, and 4.1)

Interest is computed on a *simple* basis if it is paid only on the principal of the loan. *Compound* interest is paid on accumulated loan interest as well as on the principal. Thus, if a sum of money (X_0) were borrowed at an annual interest rate i and repaid at the end of n years with accumulated interest accruing on a simple basis, the total sum repaid (FV_n or *Future Value* at the end of year n) is determined as follows:

$$FV_n = X_0(1 + ni) \tag{4.1}$$

The subscripts n and 0 merely designate time; they do not imply any arithmetic function. The product ni when multiplied by X_0 reflects the value of interest payments to be made on the loan; the value 1 accounts for the fact that the principal of the loan must be repaid. If the loan duration includes some fraction of a year, the value of n will be fractional; for example, if the loan duration were one year and three months, n would be 1.25. The total amount paid (or, the future value of the loan) will be an increasing function of the length of time the loan is outstanding (n) and the interest rate (i) charged on the loan. For example, if a consumer borrowed \$1,000 at an interest rate of 10% for one year, his total repayment would be \$1,100, determined from equation (4.1) as follows:

$$FV_1 = \$1{,}000(1 + 1 \cdot 0.1) = \$1{,}000 \cdot 1.1 = \$1{,}100.$$

If the loan were to be repaid in two years, its future value would be determined as follows:

$$FV_2 = \$1{,}000(1 + 2 \cdot 0.1) = \$1{,}000 \cdot 1.2 = \$1{,}200.$$

Continuing our example, if the loan were to be repaid in five years, its Future Value would be

$$FV_5 = \$1{,}000(1 + 5 \cdot 0.1) = \$1{,}000 \cdot 1.5 = \$1{,}500.$$

The longer the duration of a loan, the higher will be its future value. Thus, the longer lenders must wait to have their money repaid, the greater will be the total interest payments made by borrowers.

4.3 Compound Interest
(Background reading: sections 2.7, 3.1, and 4.2)

Interest is computed on a compound basis when the borrower pays interest on accumulated interest as well as on the loan principal. If interest on a given loan must accumulate for a full year before it is compounded, the future value of this loan is determined as follows:

$$FV_n = X_0(1 + i)^n. \tag{4.2}$$

For example, if an individual were to deposit \$1,000 into a savings account paying annually compounded interest at a rate of 10% (here, the bank is borrowing money), the future value of the account after five years would be \$1,610.51, determined by equation (4.2) as follows:

$$FV_5 = \$1{,}000(1 + 0.1)^5 = \$1{,}000 \cdot 1.1^5 = \$1{,}000 \cdot 1.61051 = \$1{,}610.51.$$

Notice that this sum is greater than the future value of the loan ($1,500) when interest is not compounded.

The compound interest formula can be derived intuitively from the simple interest formula. If interest must accumulate for a full year before it is compounded, then the future value of the loan after one year is $1,100, exactly the same sum as if interest had been computed on a simple basis:

$$FV_n = X_0(1 + ni) = X_0(1 + 1 \cdot i) = X_0(1 + i)^1 = \$1{,}000(1 + 0.1) = \$1{,}100. \quad (4.3)$$

The future values of loans where interest is compounded annually and when interest is computed on an annual basis will be identical only when n equals one. Since the value of this loan is $1,100 after one year and interest is to be compounded, interest and future value for the second year will be computed on the new balance of $1,100:

$$FV_2 = X_0(1 + 1 \cdot i)(1 + 1 \cdot i) = X_0(1 + i)(1 + i) = X_0(1 + i)^2,$$
$$FV_2 = \$1{,}000(1 + 0.1)(1 + 0.1) = \$1{,}000(1 + 0.1)^2 = \$1{,}210. \quad (4.4)$$

This process can be continued for five years:

$$FV_5 = \$1{,}000(1 + 0.1)(1 + 0.1)(1 + 0.1)(1 + 0.1)(1 + 0.1)$$
$$= \$1{,}000(1 + 0.1)^5 = \$1{,}610.51.$$

More generally, the process can be applied for a loan of any maturity. Therefore:

$$FV_n = X_0(1 + i)(1 + i) \cdots (1 + i) = X_0(1 + i)^n,$$
$$FV_n = \$1{,}000(1 + 0.1)(1 + 0.1) \cdots (1 + 0.1) = \$1{,}000(1 + 0.1)^n. \quad (4.5)$$

4.4 Fractional Period Compounding of Interest

In the previous examples, interest is compounded annually; that is, interest must accumulate at the stated rate i for an entire year before it can be compounded or recompounded. In many savings accounts and other investments, interest can be compounded semiannually, quarterly, or even daily. If interest is to be compounded more than once per year (or once every fractional part of a year), the future value of such an investment will be determined as follows:

$$FV_n = X_0(1 + i/m)^{mn}, \quad (4.6)$$

where interest is compounded m times per year. The interpretation of this formula is fairly straightforward. For example, if m is 2, then interest is compounded on a semiannual basis. The semiannual interest rate is simply i/m or $i/2$. If the investment is held for n periods, then it is held for $2n$ semiannual periods. Thus, we compute a semiannual interest rate $i/2$ and the number of semiannual periods the investment is held $2 \cdot n$. If \$1,000 were deposited into a savings account paying interest at an annual rate of 10% compounded semiannually, its future value after five years would be \$1,628.90, determined as follows:

$$FV_5 = \$1{,}000(1 + 0.1/2)^{2\cdot 5} = \$1{,}000(1.05)^{10} = \$1{,}000(1.62889) = \$1{,}628.90.$$

Notice that the semiannual interest rate is 5% and that the account is outstanding for ten six-month periods. This sum of \$1,628.90 exceeds the future value of the account if interest is compounded only once annually (\$1,610.51). In fact, the more times per year interest is compounded, the higher will be the future value of the account. For example, if interest on the same account were compounded monthly (12 times per year), the account's future value would be \$1,645.31:

$$FV_5 = \$1{,}000(1 + 0.1/12)^{12\cdot 5} = \$1{,}000(1.008333)^{60} = \$1{,}645.31.$$

The monthly interest rate is 0.008333 and the account is open for $m \cdot n$ or 60 months. With daily compounding, the account's value would be \$1,648.61:

$$FV_5 = \$1{,}000(1 + 0.1/365)^{365\cdot 5} = \$1{,}648.61.$$

Therefore, as m increases, future value increases, as in table 4.1. However, this rate of increase in future value becomes smaller with larger values for m; that is, the increases in FV_n induced by increases in m eventually become quite small. Thus, the difference in the future values of two accounts where interest is compounded hourly in one and every minute in the other may actually be rather trivial.

Table 4.1 Future values and annual percentage yields of accounts with initial \$10,000 deposits at 10%

Years to maturity, n	Future value simple interest (\$)	Future value compounded annually (\$)	Future value compounded monthly (\$)	Future value compounded daily (\$)	Future value compounded continuously (\$)
1	11,000	11,000	11,047	11,052	11,052
2	12,000	12,100	12,204	12,214	12,214
3	13,000	13,310	13,481	13,498	13,499
4	14,000	14,641	14,894	14,917	14,918
5	15,000	16,105	16,453	16,486	16,487
10	20,000	25,937	27,070	27,179	27,183
20	30,000	67,275	73,281	73,870	73,891
30	40,000	174,494	198,374	200,773	200,857
50	60,000	1,173,909	1,453,699	1,483,116	1,484,140
Annual percentage yield	Varies with n	0.100000	0.104713	0.1051557	0.1051709

APPLICATION 4.1: APY AND BANK ACCOUNT COMPARISONS

Financial institutions often have many ways of defining the terms or rules associated with their loans, accounts, and other investments. Such large numbers of terms and rules frequently lead to confusion among investors and consumers, particularly when trying to compare their various alternatives. For this reason, there exist several conventions which are intended to standardize the disclosure of these terms. For example, we have seen in the previous two sections the impact that changing the compounding intervals has on future value. Comparison between investments is more complicated when their numbers of compounding intervals differ. To simplify the comparison between loans with varying compounding intervals, it is often useful to compute *annual percentage yields*, also known as equivalent annual rates. The annual percentage yield (APY) represents the yield that, if compounded once per year, will produce the same future value as the stated rate i compounded m times per year:[1]

$$FV = X_0\left(1+\frac{i}{m}\right)^{mn} = X_0(1+APY)^n.$$

Thus, we can compute APY as follows:

$$APY = \left(1+\frac{i}{m}\right)^{m} - 1. \tag{4.7}$$

Because the annual percentage yield simplifies comparison between accounts with different compounding intervals, U.S. banks are normally required by law to disclose APYs along with their stated interest rates in their advertisements soliciting bank accounts. Consider an example where a savings account at bank X pays 6% interest compounded daily and a similar account at bank Y pays $6\frac{1}{4}$% interest, compounded semiannually. Which account will pay more to an investor who leaves a \$100 deposit for one year? Based on equation (4.6), we can obtain the following future values:

$$FV_X = \$100\left(1+\frac{0.06}{365}\right)^{365\cdot 1} = \$106.18313,$$

$$FV_Y = \$100\left(1+\frac{0.0625}{2}\right)^{2\cdot 1} = 106.34766.$$

Thus, an account paying a stated rate of 6% compounded daily yields a future value equivalent to an account paying slightly more than 6.18% compounded annually. An account paying a stated rate of 6.25% compounded semiannually yields a future value equivalent to an account paying slightly more than 6.437% compounded annually.

[1] If interest is not compounded, $APY = (1 + ni)^{-n} - 1$.

Therefore, the account in bank Y is preferred to that at bank X. We can arrive at the same preference ranking by examining annual percentage yields:

$$APY_X = \left(1 + \frac{0.06}{365}\right)^{365} - 1 = 0.0618313, \qquad APY_Y = \left(1 + \frac{0.0625}{2}\right)^{2} - 1 = 0.0634766.$$

Because the account at bank Y has the higher *APY*, it is preferred. The account with the higher *APY* will produce a higher future value. However, it is not necessarily true that the account with the highest stated rate also has the highest *APY*.

A 1997 advertisement in a New York newspaper offered a five-year certificate of deposit account paying interest at an annual rate of 5.83%, compounded daily. The annual percentage yield (*APY*) on this account was advertised at 6.00%. Given these details, the future value of \$100 deposited into this account can be computed to be \$133.84:

$$FV = \$100(1 + 0.0583/365)^{365\cdot 5} = \$133.84.$$

The *APY* of this account is determined as follows:

$$APY = (1 + 0.0583/365)^{365} - 1 = 0.06003.$$

The 6% *APY* advertised by the bank was approximately correct; such advertisements are often rounded slightly. In any case, the future value of this account can be determined with the 6.003% account *APY* as follows:

$$FV = \$100(1 + 0.06003)^{5} = \$133.84.$$

A \$100 initial deposit into a five-year CD account paying interest at an annual rate of 5.85%, compounded quarterly, would have a future value of \$133.69:

$$FV = \$100(1 + 0.0585/4)^{4\cdot 5} = \$133.69.$$

The *APY* of this account is 0.0598, determined as follows:

$$APY = (1 + 0.0585/4)^{4} - 1 = 0.0598.$$

Note that the future value and the *APY* of the second account are lower than those of the first account – even though the stated interest rate on the second account is higher. Compounding can have a significant effect on both future value and *APY*.

4.5 Continuous Compounding of Interest **(Background reading: sections 2.5 and 4.4)**

If interest were to be compounded an infinite number of times per period, we would say that interest is compounded continuously. However, we cannot obtain a numerical solution for future value by merely substituting in ∞ for m in equation (4.6) – calculators have no "∞" key. In the previous section, we saw that increases in m cause the future value of an investment to increase. As m approaches infinity, FV_n continues to increase, however at decreasing rates. More precisely, as m approaches infinity ($m \rightarrow \infty$), the future value of an investment can be defined as follows:

$$FV_n = X_0 e^{in}, \tag{4.8}$$

where e is the natural log whose value can be approximated at 2.718, or derived as in section 2.5.

If an investor were to deposit \$1,000 into an account paying interest at a rate of 10%, continuously compounded (or compounded an infinite number of times per year), the account's future value would be approximately \$1,648.64:

$$FV_5 = \$1{,}000 \cdot e^{1 \cdot 5} \approx \$1{,}000 \cdot 2.718^{0.5} = \$1{,}648.64.$$

The Future Value of this account exceeds only slightly the value of the account if interest were compounded daily. Also, note that continuous compounding simply means that interest is compounded an infinite number of times per time period.

4.6 Annuity Future Values
(Background reading: sections 2.8, 3.4, and 4.3)

An *annuity* is a series of equal payments made at equal intervals. Suppose that payments are to be made into an interest-bearing account. The future value of that account will be a function of interest accruing on prior deposits as well as the deposits themselves. A future value annuity factor (fvaf) is used to determine the future value of an annuity. This annuity is a series of equal payments made at identical intervals. The future value annuity factor may be derived through the use of the geometric expansion procedure discussed in section 3.4. This technique is very useful for future value computations when a large number of time periods are involved. The geometric expansion enables us to reduce a repetitive expression requiring many calculations to an expression that can be computed much more quickly. Suppose that we wish to determine the future value of an account based on a payment of X made at the end of each year t for n years, where that account pays an annual interest rate equal to i:

$$FVA = X[(1+i)^{n-1} + (1+i)^{n-2} + \ldots + (1+i)^2 + (1+i)^1 + 1]. \tag{4.9}$$

The payment made at the end of the first year will accumulate interest for a total of $n - 1$ years, the payment at the end of the second year will accumulate interest for $n - 2$ years, and so on. Clearly, determining the future value of this account with equation (4.9) will be very time-consuming if n is large. The first step in the geometric expansion to simplify equation (4.9) is to multiply both of its sides by $1 + i$:

$$FVA(1+i) = X[(1+i)^n + (1+i)^{n-1} + \ldots + (1+i)^3 + (1+i)^2 + (1+i)]. \quad (4.10)$$

The second step in this geometric expansion is to subtract equation (4.9) from equation (4.10), to obtain:

$$FVA(1+i) - FVA = X[(1+i)^n - 1]. \quad (4.11)$$

Notice that the subtraction led to the cancellation of many terms, reducing the equation that we wish to compute with to a much more manageable size. Finally, we rearrange terms in equation (4.12) to obtain equations (4.12) and (4.13):

$$FVA \cdot 1 + FVA \cdot i - FVA = X[(1+i)^n - 1] = FVA \cdot i = X[(1+i)^n - 1], \quad (4.12)$$

$$FVA = \frac{X[(1+i)^n - 1]}{i}. \quad (4.13)$$

Practicing derivations such as this is an excellent way to understand the intuition behind financial formulas. Understanding the derivations is necessary in order to be able to modify the formulas for a variety of more complex (and realistic) scenarios.

Consider an example applicable to many individuals who open Individual Retirement Accounts (I.R.A.'s), from which they may withdraw when they reach the age of $59\frac{1}{2}$ years. Consider an individual who makes a \$2,000 contribution to his I.R.A. at the end of each year for 20 years. All of his contributions receive a 10% annual rate of interest, compounded annually. What will be the total value of this account, including accumulated interest, at the end of the 20-year period? Equation (4.13) can be used to evaluate the future value of this annuity, where X is the annual contribution made at the end of each year by the investor to his account, i is the interest rate on the account, and FVA is the future value of the annuity. The future value of this individual's I.R.A. is \$114,550:

$$FVA_n = \$2{,}000\frac{(1+0.10)^{20} - 1}{0.10} = \$114{,}550.$$

This future value annuity equation can be used whenever identical periodic contributions are made toward an account. Section 4.8 will present a discussion on determining the present value of such a series of cash flows. (The term "present value" is also defined later in section 4.8.)

Note that each of the above calculations assumes that cash flows are paid at the end of each period. If, instead, cash flows were realized at the beginning of each period, the annuity would be referred to as an *annuity due*. The annuity due would generate an extra year of interest on each cash flow. Hence, the future value of an annuity due is determined by simply multiplying the future value annuity formula by $(1 + i)$:

$$FVA_{n,\text{due}} = X\frac{(1+i)^n - 1}{i}(1+i) = X\frac{(1+i)^{n+1} - (1+i)}{i}. \quad (4.14)$$

From the above example, we find that the future value of the individual's I.R.A. is \$126,005 if payments to the I.R.A. are made at the beginning of each year:

$$FVA_{n,\text{due}} = \$2{,}000\frac{(1+0.10)^{21}-(1+0.10)}{0.10} = \$126{,}005.$$

APPLICATION 4.2: PLANNING FOR RETIREMENT
(Background reading: sections 2.5, 3.1, and 4.5)

Suppose that a 23-year-old accountant wishes to retire as a millionaire based on her retirement savings account. She intends to open and contribute to a tax-deferred 401k retirement account sponsored by her employer each year until she retires with \$1,000,000 in that account. Would she meet her retirement goal if she deposited \$10,000 into that account at the end of each year until she is 65 years of age? Assume that her account will generate an annual rate of interest equal to 5% for each of the next 42 years.

Equation (4.13) will be used to solve this problem:

$$FV_n = X\frac{(1+i)^n-1}{i} = \$10{,}000\cdot\frac{(1+0.05)^{42}-1}{0.05}$$

$$= \$10{,}000\cdot\frac{7.76159-1}{0.05} = \$1{,}352{,}318.$$

Now, suppose that she would like to retire as soon as possible with \$1,000,000 in her account. Assuming that nothing else associated with her situation changes, what is the earliest age at which she can retire?

Now, we will use equation (4.13) to algebraically solve for n, the number of years that the accountant must wait to retire:

$$FV_n = X\frac{(1+i)^n-1}{i}, \qquad \frac{FV_n\cdot i}{X} = (1+i)^n - 1,$$

$$\log\left(\frac{FV_n\cdot i}{X}+1\right) = n\cdot\log(1+i), \qquad \log\left(\frac{FV_n\cdot i}{X}+1\right)\div\log(1+i) = n,$$

$$n = \log\left(\frac{\$1{,}000{,}000\cdot 0.05}{\$10{,}000}+1\right)\div\log(1+0.05) = \frac{\log(6)}{\log(1.05)} = 36.72.$$

Since payments are made at the end of each year, the accountant must wait 37 years when she is 60 before she can retire as a millionaire. Note that we were able to find a closed-form solution (put n on one side alone) using simple algebra. In many time value problems, the exact placement of the exponent n will prevent us from obtaining a solution so easily.

4.7 Discounting and Present Value
(Background reading: section 4.3)

Cash flows realized at the present time have a greater value to investors than cash flows realized later, for the following reasons:

1 *Inflation*. The purchasing power of money tends to decline over time.
2 *Risk*. We never know with certainty whether we will actually realize the cash flow that we are expecting.
3 The option to either spend money now or defer spending it is likely to be worth more than being forced to defer spending the money.

The purpose of the Present-Value model is to express the value of a future cash flow in terms of cash flows at present. Thus, the Present-Value model is used to compute how much an investor would pay now for the expectation of some cash flow to be received in n years. The present value of this cash flow would be a function of inflation, the length of wait before the cash flow is received (n), the riskiness associated with the cash flow, and the time value an investor associates with money (how much he needs money now as opposed to later). Perhaps the easiest way to account for these factors when evaluating a future cash flow is to discount it in the following manner:

$$PV = \frac{CF_n}{(1+k)^n}, \tag{4.15}$$

where CF_n is the cash flow to be received in year n, k is an appropriate discount rate accounting for risk, inflation, and the investor's time value associated with money, and PV is the present value of that cash flow. The discount rate enables us to evaluate a future cash flow in terms of cash flows realized today. Thus, the maximum a rational investor would be willing to pay for an investment yielding a \$9,000 cash flow in six years assuming a discount rate of 15% would be \$3,891, determined as follows:

$$PV = \frac{\$9{,}000}{(1+0.15)^6} = \frac{\$9{,}000}{2.31306} = \$3{,}890.95.$$

In the above example, we simply assumed a 15% discount rate. Realistically, perhaps the easiest value to substitute for k is the current interest or return rate on loans or other investments of similar duration and riskiness. However, this market-determined interest rate may not consider the individual investor's time preferences for money. Furthermore, the investor may find difficulty in locating a loan (or other investment) of similar duration and riskiness. For these reasons, more scientific methods for determining appropriate discount rates will be discussed later. In any case, the discount rate should account for inflation, the riskiness of the investment, and the investor's time value for money.

Deriving the present-value formula

The present-value formula can be derived easily from the compound interest formula. Assume that an investor wishes to deposit a sum of money into a savings account paying interest at a rate of 15%, compounded annually. If the investor wishes to withdraw from his account \$9,000 in six years, how much must he deposit now? This answer can be determined by solving the compound interest formula for X_0:

$$FV_n = X_0(1+i)^n, \qquad X_0 = \frac{FV_n}{(1+i)^n} = \frac{\$9{,}000}{(1+0.15)^6} = \frac{\$9{,}000}{2.31306} = \$3{,}890.95.$$

Therefore, the investor must deposit \$3,890.95 now in order to withdraw \$9,000 in six years at 15%.

Notice that the present-value equation (4.15) is almost identical to the compound interest formula where we solve for the principal (X_0):

$$PV = \frac{CF_n}{(1+k)^n}, \qquad X_0 = \frac{FV_n}{(1+i)^n}.$$

Mathematically, these formulas are the same; however, there are some differences in their economic interpretations. In the interest formulas, interest rates are determined by market supply and demand conditions, whereas discount rates are individually determined by investors themselves (although their calculations may be influenced by market interest rates). In the present-value formula, we wish to determine how much some future cash flow is worth now; in the interest formula above, we wish to determine how much money must be deposited now to attain some given future value.

4.8 The Present Value of a Series of Cash Flows (Background reading: sections 2.8 and 4.7)

Suppose that an investor needs to evaluate a series of cash flows. She needs only to discount each separately and then sum the present values of each of the individual cash flows. Thus, the present value of a series of cash flows CF_t received in time period t can be determined by the following expression:

$$PV = \sum_{t=1}^{n} \frac{CF_t}{(1+k)^t}. \tag{4.16}$$

For example, if an investment were expected to yield annual cash flows of \$200 for each of the next five years, assuming a discount rate of 5%, its present value would be \$865.90:

$$PV = \frac{200}{(1+0.05)^1} + \frac{200}{(1+0.05)^2} + \frac{200}{(1+0.05)^3} + \frac{200}{(1+0.05)^4} + \frac{200}{(1+0.05)^5} = \$865.90.$$

Therefore, the maximum price an individual should pay for this investment is \$865.90, even though the cash flows yielded by the investment total \$1,000. Because the individual must wait up to five years before receiving the \$1,000, the investment is worth only \$865.90. Use of the present-value series formula does not require that cash flows CF_t in each year be identical, as does the annuity model presented in the next section.

4.9 Annuity Present Values
(Background reading: sections 3.4, 4.6, and 4.8)

The expression for determining the present value of a series of cash flows can be quite cumbersome, particularly when the payments extend over a long period of time. This formula requires that n cash flows be discounted separately and then summed. When n is large, this task may be rather time-consuming. If the annual cash flows are identical and are to be discounted at the same rate, an annuity formula can be a useful time-saving device. The same problem as discussed in the previous section can be solved using the following annuity formula:

$$PV_{\mathrm{A}} = \frac{CF}{k}\left[1 - \frac{1}{(1+k)^n}\right], \tag{4.17}$$

where CF is the level of the annual cash flow generated by the annuity (or series). Use of this formula does require that all of the annual cash flows be identical. Thus, the present value of the cash flows in the problem discussed in the previous section is \$865.90, determined as follows:

$$PV_{\mathrm{A}} = \frac{\$200}{0.05}\left[1 - \frac{1}{(1+0.05)^5}\right] = \$4{,}000(0.2164738) = \$865.90.$$

As n becomes larger, this formula becomes more useful relative to the present-value series formula discussed in the previous section. However, the annuity formula requires that all cash flows be identical and be paid at the end of each year. The present-value annuity formula can be derived easily from the perpetuity formula discussed in section 4.11, or from the geometric expansion procedure described later in this section.

Note that each of the above calculations assumes that cash flows are paid at the end of each period. If, instead, cash flows were realized at the beginning of each period, the annuity would be referred to as an annuity due. Each cash flow generated by the annuity due would, in effect, be received one year earlier than if cash flows were realized at the end of each year. Hence, the present value of an annuity due is determined by simply multiplying the present-value annuity formula by $(1 + k)$:

$$PVA_{\mathrm{due}} = \frac{CF}{k}\left[1 - \frac{1}{(1+k)^n}\right](1+k). \tag{4.18}$$

The present value of the five-year annuity due discounted at 5% is determined as follows:

$$PVA_{\text{due}} = \frac{200}{0.05}\left[1 - \frac{1}{(1+0.05)^5}\right](1+0.05) = \$4{,}000[0.2164738](1.05) = 909.19.$$

Deriving the present-value annuity formula

The present value annuity factor (pvaf) may be derived through use of the geometric expansion (see section 3.4). Consider the case where we wish to determine the present value of an investment based on a cash flow of *CF* made at the end of each year *t* for *n* years, where the appropriate discount rate is *k*:

$$PV_{\text{A}} = CF \cdot \left[\frac{1}{(1+k)^1} + \frac{1}{(1+k)^2} + \ldots + \frac{1}{(1+k)^n}\right]. \tag{A}$$

Thus, the payment made at the end of the first year is discounted for one year, the payment at the end of the second year is discounted for two years, and so on. Clearly, determining the present value of this account will be very time-consuming if n is large. The first step of the geometric expansion is to multiply both sides of (A) by $(1 + k)$:

$$PV_{\text{A}}(1+k) = CF \cdot \left[1 + \frac{1}{(1+k)^1} + \ldots + \frac{1}{(1+k)^{n-1}}\right]. \tag{B}$$

The second step in the geometric expansion is to subtract equation (A) from equation (B), to obtain:

$$PV_{\text{A}}(1+k) - PV_{\text{A}} = CF\left[1 - \frac{1}{(1+k)^n}\right], \tag{C}$$

which simplifies to

$$PV_{\text{A}}(1+k-1) = PV_{\text{A}}(k) = CF\left[1 - \frac{1}{(1+k)^n}\right]. \tag{D}$$

Notice that the subtraction led to the cancellation of many terms, reducing the equation that we wish to compute to a much more manageable size. Finally, we cancel the ones on the left side and divide both sides of equation (D) by k, to obtain:

$$PV_{\text{A}} = \frac{CF}{k}\left[1 - \frac{1}{(1+k)^n}\right]. \tag{4.17}$$

APPLICATION 4.3: PLANNING FOR RETIREMENT, PART II (Background reading: application 4.2 and section 4.9)

Suppose that the 23-year-old accountant from application 4.2 wishes to retire as a millionaire based on her retirement savings account, but needs to know what the present value of that million-dollar account is. If the account is open for the full 37 years, its future value will be \$1,016,282, based on equation (4.13). Based on a discount rate of 5% and assuming that the account is open for 37 years, its present value is easily determined from equation (4.15) as follows:

$$PV = \frac{CF_n}{(1+k)^n} = \frac{\$1{,}016{,}282}{(1+0.05)^{37}} = \$167{,}112.07.$$

In present-value terms, this million-dollar account is obviously worth much less than \$1,000,000. However, what is the present value of the annual series \$10,000 deposits that she will make to that account? Again, based on a 5% discount rate, we determine this present value with equation (4.17) as follows:

$$PV_{\mathrm{A}} = \frac{\$10{,}000}{0.05}\left[1 - \frac{1}{(1+0.05)^{37}}\right] = \$167{,}112.97.$$

Notice that the present value of contributions that she makes to the account is identical to the present value of what she will be able to retire with.

APPLICATION 4.4: VALUING A BOND

Because the present value of a series of cash flows is simply the sum of the present values of the cash flows, the annuity formula can be combined with other present-value formulas to evaluate investments. Consider, for example, a 7% coupon bond making annual interest payments for nine years. If this bond has a \$1,000 face (or par) value, and its cash flows are discounted at 6%, its cash flows will be \$70 in each of the nine years plus \$1,000 in the tenth year. The present value of the bond's cash flows can be determined as follows:

$$PV = \frac{\$70}{0.06}\left[1 - \frac{1}{(1+0.06)^9}\right] + \frac{\$1{,}000}{(1+0.06)^9} = \$1{,}166.67(0.4081015) + \frac{\$1{,}000}{1.689479}$$

$$= \$476.118 + 591.898 = \$1{,}068.017.$$

Thus, the value of a bond is simply the sum of the present values of the cash flow streams resulting from interest payments and from principal repayment.

Now, let us revise the above example to value another 7% coupon bond. This bond will make semiannual (twice yearly) interest payments for nine years. If this bond has a \$1,000 face (or par) value, and its cash flows are discounted at the stated annual rate of 6%, its value can be determined as follows:

$$PV = \frac{\$35}{0.03}\left[1 - \frac{1}{(1+0.03)^{18}}\right] + \frac{\$1{,}000}{(1+0.03)^{18}} = \$1{,}166.67(0.4126) + \frac{\$1{,}000}{1.7024}$$

$$= \$481.373 + 587.395 = \$1{,}068.768.$$

Again, the value of the bond is the sum of the present values of the cash flow streams resulting from interest payments and from the principal repayment. However, the semiannual discount rate equals 3% and payments are made to bondholders in each of 18 semiannual periods.

4.10 Amortization
(Background reading: section 4.9)

At the beginning of this chapter, we derived the concept of present value from that of future value. Amortization is essentially a topic relating to interest, but the present-value annuity model presented in this chapter is crucial to its development. Amortization is the payment structure associated with a loan. That is, the amortization schedule of a loan is its payment schedule. Consider the annuity model from equation (4.17):

$$PV_A = \frac{CF}{k}\left[1 - \frac{1}{(1+k)^n}\right]. \tag{4.17}$$

Typically, when a loan is amortized, the loan repayments will be made in equal amounts; that is, each annual or monthly payment will be identical. At the end of the repayment period, the balance (amount of principal remaining) on the loan will be zero. Thus, each payment made by the borrower is applied to the principal repayment as well as to interest. A bank lending money will require that the sum of the present values of its repayments be at least as large as the sum of money it loans. Therefore, if the bank loans a sum of money equal to *PV* for *n* years at an interest rate of *i*, the amount of the annual loan repayment will be *CF*:

$$CF = [PV_A \cdot k] \div \left[1 - \frac{1}{(1+k)^n}\right]. \tag{4.18}$$

For example, if a bank were to extend a \$865,895 five year mortgage to a corporation at an interest rate of 5%, the corporation's annual payment on the mortgage would be \$200,000, determined by equation (4.18):

Table 4.2 The amortization schedule of a $865,895 loan with equal annual payments for five years at 5%

Year	Principal ($)	Payment ($)	Interest ($)	Payment to principal ($)
1	865,895	200,000	43,295	156,705
2	709,189	200,000	35,459	164,541
3	544,649	200,000	27,232	172,768
4	371,881	200,000	18,594	181,406
5	190,476	200,000	9,524	190,476

The loan is fully repaid by the end of the fifth year. The principal represents the balance at the beginning of the given year. The payment is made at the end of the given year, and includes one year of interest accruing on the principal from the beginning of that year. The remaining part of the payment is payment to the principal. This payment to the principal is deducted from the principal or balance as of the beginning of the following year.

$$CF = [\$865{,}895 \cdot 0.05] \div \left[1 - \frac{1}{(1+0.05)^5}\right] = \$200{,}000.$$

Thus, each year, the corporation will pay $200,000 toward both the loan principal and interest obligations. The amounts attributed to each are given in table 4.2. Notice that as payments are applied toward the principal, the principal declines; correspondingly, the interest payments decline. Nonetheless, total annual payments are identical until the principal diminishes to zero in the fifth year.

APPLICATION 4.5: DETERMINING THE MORTGAGE PAYMENT

A family has purchased a home with $30,000 down and a $300,000 mortgage. The mortgage will be amortized over 30 years with equal monthly payments. The interest rate on the mortgage will be 9% per year. Based on this data, we would like to determine the monthly mortgage payment and compile an amortization table decomposing each of the monthly payments into interest and payment toward principle.

First, we will express annual data as monthly data. Three hundred and sixty ($12 \cdot 30$) months will elapse before the mortgage is fully paid, and the monthly interest rate will be 0.0075, or 9% divided by 12. Given this monthly data, monthly mortgage payments are determined as follows:

$$Payment = \$300{,}000 \div \left(\frac{1}{0.0075} - \frac{1}{0.0075(1+0.0075)^{360}}\right) = \$2{,}413.87.$$

Table 4.3 depicts the amortization schedule for this mortgage.

Table 4.3 The amortization schedule of a $300,000 loan with equal monthly payments for 30 years at 9% interest per annum (0.0075% per month)

Month	Beginning-of-month principal ($)	Total payment ($)	Payment on interest ($)	Payment on principal ($)
1	300,000.00	2,413.87	2,250.00	163.87
2	299,836.13	2,413.87	2,248.77	165.10
3	299,671.03	2,413.87	2,247.53	166.34
4	299,504.69	2,413.87	2,246.29	167.58
5	299,337.11	2,413.87	2,245.03	168.84
⋮	⋮	⋮	⋮	⋮
358	7,134.33	2,413.87	53.51	2,360.36
359	4,773.97	2,413.87	35.80	2,378.07
360	2,395.90	2,413.87	17.97	2,395.90

Students should be able to work through the figures on this table starting from the upper left-hand corner, then working to the left, then down. In this particular example, because n is large (360), use of a computerized spreadsheet will make computations substantially more efficient.

4.11 Perpetuity Models (Background reading: section 4.9)

As the value of n approaches infinity in the annuity formula, the value of the right-hand side term in the brackets,

$$\frac{1}{(1+k)^n},$$

approaches zero. That is, the cash flows associated with the annuity are paid each year for a period approaching "forever." Therefore, as n approaches infinity, the value of the infinite time horizon annuity approaches

$$PV_A = \frac{CF}{k}[1-0];$$

$$PV_P = \frac{CF}{k}. \tag{4.19}$$

The annuity formula discussed in section 4.9 can be derived intuitively by use of figure 4.1. First, consider a perpetuity as a series of cash flows beginning at time period one (one year from now) and extending indefinitely into perpetuity. Consider a second perpetuity with cash flows beginning in time period n and extending indefinitely into perpetuity. If an investor is to receive an n-year annuity, the second perpetuity represents those cash flows from the first perpetuity that he will not receive. Thus, the

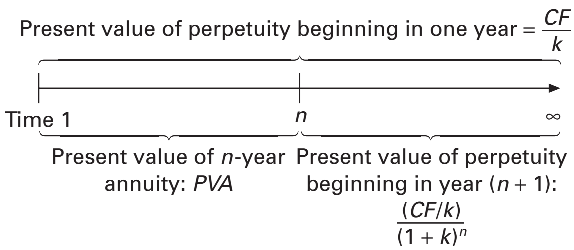

Figure 4.1 Deriving annuity present value from perpetuity present values. The present value of a perpetuity beginning in one year minus the present value of a second perpetuity beginning in year $(n + 1)$ equals the present value of an n-year annuity. Thus, $PVA = CF/k - (CF/k) \div (1 + k)^n = CF/k \cdot [1 - 1/(1 + k)^n]$.

difference between the present values of the first and second perpetuities represents the value of the annuity that he will receive. Note that the second perpetuity is discounted a second time, since its cash flows do not begin until year n:

$$PV_{\mathrm{A}} = \frac{CF}{k} - \frac{CF/k}{(1+k)^n} = \frac{CF}{k}\left[1 - \frac{1}{(1+k)^n}\right].$$

The perpetuity model is useful in the evaluation of a number of investments. Any investment with an indefinite or perpetual life expectancy can be evaluated with the perpetuity model. For example, the present value of a stock, if its dividend payments are projected to be stable, will be equal to the amount of the annual dividend (cash flow) generated by the stock divided by an appropriate discount rate. In European financial markets, a number of perpetual bonds have been traded for several centuries. In many regions in the United States, ground rents (perpetual leases on land) are traded. The proper evaluation of these and many other investments requires the use of perpetuity models.

The maximum price an investor would be willing to pay for a perpetual bond generating an annual cash flow of \$200, each discounted at a rate of 5%, can be determined from equation (4.19):

$$PV_{\mathrm{P}} = \frac{\$200}{0.05} = \$4{,}000.$$

4.12 Single-Stage Growth Models
(Background reading: sections 4.9 and 4.11)

If the cash flow associated with an investment were expected to grow at a constant annual rate of g, the amount of the cash flow generated by that investment in year t would be

$$CF_t = CF_1(1 + g)^{t-1}, \tag{4.20}$$

where CF_1 is the cash flow generated by the investment in year one. Thus, if a stock paying a dividend of \$100 in year one were expected to increase its dividend payment by 10% each year thereafter, the dividend payment in the fourth year would be \$133.10:

$$CF_4 = CF_1(1 + 0.10)^{4-1}$$

Similarly, the cash flow generated by the investment in the following year ($t + 1$) will be

$$CF_{t+1} = CF_1(1 + g)^t.$$

The stock's dividend in the fifth year will be \$146.41:

$$CF_{4+1} = CF_1(1 + 0.10)^4 = \$146.41.$$

If the stock had an infinite life expectancy (as most stocks might be expected to), and its dividend payments were discounted at a rate of 13%, the value of the stock would be determined by

$$PV_{\text{gp}} = \frac{\$100}{0.13 - 0.10} = \frac{\$100}{0.03} = \$3{,}333.33.$$

This expression is often called the Gordon Stock Pricing Model. It assumes that the cash flows (dividends) associated with the stock are known in the first period and will grow at a constant compound rate in subsequent periods. More generally, this growing perpetuity expression can be written as follows:

$$PV_{\text{gp}} = \frac{CF_1}{k - g}. \tag{4.21}$$

The growing perpetuity expression simply subtracts the growth rate from the discount rate; the growth in cash flows helps to "cover" the time value of money. This formula for evaluating growing perpetuities can be used only when $k > g$. If $g > K$, either the growth rate or discount rate has probably been calculated improperly. Otherwise, the investment would have an infinite value (even though the formula would generate a negative value).

The formula (4.22) for evaluating growing annuities can be derived intuitively from the growing perpetuity model. In figure 4.2, the difference between the present value of a growing perpetuity with cash flows beginning in time period n is deducted from the present value of a perpetuity with cash flows beginning in year one, resulting in the present value of an n-year growing annuity. Notice that the amount of the cash flow generated by the growing annuity in year ($n + 1$) is $CF(1 + g)^n$. This is the first of the

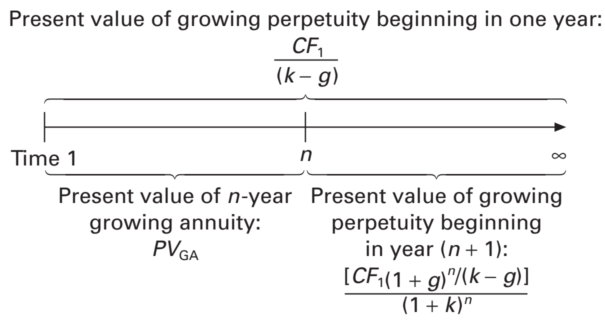

Figure 4.2 Deriving growing annuity present value from growing perpetuity peresent value. The present value of a growing perpetuity beginning in one year minus the present value of a second growing perpetuity beginning in year $(n + 1)$ equals the present value of an n-year growing annuity: $CF_1/(k - g) \div [CF_1/(k - g)] - (1 + k)^n$; $PV_{GA} = [CF_1/(k - g)][1 \div (1 + g)^n] - (1 + k)^n$.

cash flows not generated by the growing annuity; it is generated after the annuity is sold or terminated. Because the cash flow is growing at the rate g, the initial amount of the cash flow generated by the second perpetuity is exceeded by the initial cash flow of the perpetuity beginning in year one:

$$PV_{GA} = \frac{CF_1}{k-g} \cdot \left[1 - \frac{(1+g)^n}{(1+k)^n}\right]. \tag{4.22}$$

Cash flows generated by many investments will grow at the rate of inflation. For example, consider a project undertaken by a corporation whose cash flow in year one is expected to be \$10,000. If cash flows were expected to grow at the inflation rate of 6% each year until year six, then terminate, the project's present value would be \$48,320.35, assuming a discount rate of 11%:

$$PV_{GA} = \frac{\$10{,}000}{0.11-0.06}\left[1 - \frac{(1+0.06)^6}{(1+0.11)^6}\right] = \$200{,}000(1-0.7584) = \$48{,}320.35.$$

Cash flows are generated by this investment through the end of the sixth year. No cash flow was generated in the seventh year. Verify that the amount of cash flow that would have been generated by the investment in the seventh year if it had continued to grow would have been $\$10{,}000(1.06)^6 = \$14{,}185$.

APPLICATION 4.6: STOCK VALUATION MODELS

Consider a stock whose annual dividend next year is projected to be \$50. This payment is expected to grow at an annual rate of 5% in subsequent years. An investor has determined that the appropriate discount rate for this stock is 10%. The current value of this stock is \$1,000, determined by the growing perpetuity model:

$$PV_{\text{gp}} = \frac{\$50}{0.10 - 0.05} = \$1{,}000.$$

This model is often referred to as the Gordon Stock Pricing Model. It may seem that this model assumes that the stock will be held by the investor forever. But what if the investor intends to sell the stock in five years? Its value would be determined by the sum of the present values of cash flows the investor does expect to receive:

$$PV_{\text{GA}} = \frac{DIV_1}{k - g} \cdot \left[1 - \frac{(1+g)^n}{(1+k)^n}\right],$$

where P_n is the price the investor expects to receive when he sells the stock in year n; and DIV_1 is the dividend payment the investor expects to receive in year one. The present value of the dividends the investor expects to receive is \$207.53:

$$PV_{\text{GA}} = \frac{\$50}{0.10 - 0.05}\left[1 - \frac{(1+0.05)^5}{(1+0.10)^5}\right] = \$207.53.$$

The selling price of the stock in year five will be a function of the dividend payments that the prospective purchaser expects to receive beginning in year six. Thus, in year five, the prospective purchaser will pay \$1,276.28 for the stock, based on his initial dividend payment of \$63.81, determined by the following equations:

$$DIV_6 = DIV_1(1 + 0.05)^{6-1} = \$63.81,$$

$$\text{stock value in year five} = 63.81/(0.10 - 0.05) = \$1{,}276.28.$$

The present value of the \$1,276.28 that the investor will receive when he sells the stock at the end of the fifth year is \$792.47:

$$PV = \frac{\$1{,}276.28}{(1+0.1)^5} = \$792.47.$$

The total stock value will be the sum of the present values of the dividends received by the investor and his cash flows received from the sale of the stock. Thus, the current value of the stock is \$207.53 plus \$792.47, or \$1,000. This is exactly the same sum determined by the growing perpetuity model earlier; therefore, the growing perpetuity model can be used to evaluate a stock even when the investor expects to sell it.

4.13 Multiple-Stage Growth Models
(Background reading: section 4.12)

The Gordon Stock Pricing Model may be unrealistic in many scenarios, in that it assumes that one growth rate applies to the firm's cash flows and that this growth rate extends forever. Multiple-stage growth models enable the user to allow for different growth rates in different periods. For example, a growth company might generate cash flows that are expected to grow at a high rate in the short term and then decline as the firm matures. The multistage growth model can accommodate this pattern.

Suppose, for example, that an investor has the opportunity to invest in a stock currently selling for $100 per share. The stock is expected to pay a $3 dividend next year (at the end of year 1). In each subsequent year until the seventh year, the annual dividend is expected to grow at a rate of 20%. Starting in the eighth year, the annual dividend will grow at an annual rate of 3% forever. All cash flows are to be discounted at an annual rate of 10%. Should the stock be purchased at its current price?

The following Two-Stage Growth Model can be used to evaluate this stock:

$$P_0 = DIV_1\left[\frac{1}{k-g_1} - \frac{(1+g_1)^n}{(k-g_1)(1+k)^n}\right] + \frac{DIV_1(1+g_1)^{n-1}(1+g_2)}{(k-g_2)(1+k)^n}. \tag{4.23}$$

Note that this model begins with an n-year annuity at growth rate g_1 and accommodates the new growth rate g_2 in the growing perpetuity that follows. The perpetuity is discounted a second time because it is deferred; it does not commence payments until year n. Substituting values from the problem statement yields the following:

$$P_0 = \$3\left[\frac{1}{0.1-0.2} - \frac{(1+0.2)^7}{(0.1-0.2)(1+0.1)^7}\right] + \frac{\$3(1+0.2)^{7-1}(1+0.03)}{(0.1-0.03)(1+0.1)^7} = 92.8014519.$$

Since the $100 purchase price of the stock exceeds its 92.8014519 value, the stock should not be purchased.

The following represents a Three-Stage Growth Model which is based on a growing annuity, a deferred growing annuity, and a deferred growing perpetuity:

$$P_0 = DIV_1\left[\frac{1}{k-g_1} - \frac{(1+g_1)^{n(1)}}{(k-g_1)(1+k)^{n(1)}}\right]$$
$$+ DIV_1\left[\frac{(1+g_1)^{n(1)-1}(1+g_2)}{(1+k)^{n(1)}(k-g_2)} - \frac{(1+g_1)^{n(1)-1}(1+g_2)^{n(2)-n(1)+1}}{(k-g_2)(1+k)^{n(2)}}\right]$$
$$+ \frac{DIV_1(1+g_1)^{n(1)-1}(1+g_2)^{n(2)-n(1)}(1+g_3)}{(k-g_3)(1+k)^{n(2)}}.$$

There are three stages here, the first ending at time $n(1)$, the second ending at time $n(2)$, and the third extending into perpetuity. It may be a useful exercise to closely examine this expression to determine why it is structured in this manner. Try to determine why the growth rates and discount rates are structured as they are. Be certain to first be comfortable with the Present-Value Growing Annuity and Perpetuity Models and the Two-Stage Growth Model.

EXERCISES

4.1. The Ruth Company borrowed \$21,000 at an annual interest rate of 9%. What is the future value of this loan assuming interest is accumulated on a simple basis?

4.2. The Cobb Company has issued ten million dollars in 10% coupon bonds maturing in five years. Interest payments on these bonds will be made semiannually.

(a) How much are Cobb's semiannual interest payments?
(b) What will be the total payment made by Cobb on the bonds in each of the first four years?
(c) What will be the total payment made by Cobb on the bonds in the fifth year?

4.3. I have the opportunity to deposit \$10,000 into my savings account today, which pays interest at an annual rate of 5.5%, compounded daily. What will be the ending balance of my account in five years if I make no additional deposits or withdrawals?

4.4. What would be the future value of the loan in problem 4.1 if interest were compounded:

(a) annually?
(b) semiannually?
(c) monthly?
(d) daily?
(e) continuously?

4.5. A consumer has the opportunity to deposit \$10,000 into his savings account today, which pays interest at an annual rate of 5.5%, compounded daily. What will be the ending balance of his account in five years if he makes no additional deposits or withdrawals?

4.6. The Speaker Company has the opportunity to purchase a five-year $1,000 certificate of deposit (CD) paying interest at an annual rate of 12%, compounded annually. The company will not withdraw early any of the money in its CD account. Will this account have a greater future value than a five-year $1,000 CD paying an annual interest rate of 10%, compounded daily?

4.7. The Waner Company needs to set aside a sum of money today for the purpose of purchasing, for $10,000, a new machine in three years. Money used to finance this purchase will be placed in a savings account paying interest at a rate of 8%. How much money must be placed in this account now to assure the Waner company $10,000 in three years if interest is compounded yearly?

4.8. A given savings account pays interest at an annual rate of 3% compounded quarterly. Find the annual percentage yield (*APY*) for this account.

4.9.* Assuming no withdrawals or additional deposits, how much time is required for $1,000 to double if placed in a savings account paying an annual interest rate of 10% if interest were:

(a) computed on a simple basis?
(b) compounded annually?
(c) compounded monthly?
(d) compounded continuously?

4.10. What is the present value of a security promising to pay $10,000 in five years if its associated discount rate is:

(a) 20%?
(b) 10%?
(c) 1%?
(d) 0%?

4.11. What is the present value of a security to be discounted at a 10% rate promising to pay $10,000 in:

(a) 20 years?
(b) ten years?
(c) one year?
(d) six months?
(e) 73 days?

4.12. The Gehrig Company is considering an investment that will result in a $2,000 cash flow in one year, a $3,000 cash flow in two years, and a $7,000 cash flow in three years. What is the present value of this investment if all cash flows are to be discounted at an 8% rate? Should Gehrig Company management be willing to pay $10,000 for this investment?

4.13. The Cramden Company has the opportunity to pay $30,000 for a security which promises to pay $6,000 in each of the next nine years. Would this be a wise investment if the appropriate discount rate were:

(a) 5%?
(b) 10%?
(c) 20%?

4.14. The Larsen Company is selling preferred stock which is expected to pay a $50 annual dividend per share. What is the present value of dividends associated with each share of stock if the appropriate discount rate were 8% and its life expectancy were infinite?

4.15. The Dinkins Company has purchased a machine whose output will result in a $5,000 cash flow in its first year of operation. This cash flow is projected to grow at the annual 10% rate of inflation over each of the next ten years. What will be the cash flow generated by this machine in:

(a) its second year of operation?
(b) its third year of operation?
(c) its fifth year of operation?
(d) its tenth year of operation?

4.16. The Wagner Company is considering the purchase of an asset that will result in a $5,000 cash flow in its first year of operation. Annual cash flows are projected to grow at the 10% annual rate of inflation in subsequent years. The life expectancy of this asset is seven years, and the appropriate discount rate for all cash flows is 12%. What is the maximum price that Wagner should be willing to pay for this asset?

4.17. What is the present value of a stock whose $100 dividend payment next year is projected to grow at an annual rate of 5%? Assume an infinite life expectancy and a 12% discount rate.

4.18. Which of the following series of cash flows has the highest present value at a 5% discount rate:

(a) $500,000 now?
(b) $100,000 per year for eight years?
(c) $60,000 per year for 20 years?
(d) $30,000 each year forever?

4.19. Which of the cash flow series in problem 4.18 has the highest present value at a 20% discount rate?

4.20. Mr. Sisler has purchased a $200,000 home with $50,000 down and a 20-year mortgage at a 10% interest rate. What will be the periodic payment on this mortgage if they are made:

(a) annually?
(b) monthly?

4.21. What discount rate for cash flows in problem 4.13 would render the Cramden Company indifferent regarding its decision to invest $30,000 for the nine-year series of $6,000 cash flows? That is, what discount rate will result in a $30,000 present value for the series?

4.22.* What would be the present value of $10,000 to be received in 20 years if the appropriate discount rate of 10% were compounded:

(a) annually?
(b) monthly?
(c) daily?
(d) continuously?

4.23. (a) What would be the present value of a 30-year annuity if the $1,000 periodic cash flow were paid monthly? Assume a discount rate of 10% per year.
(b) Should an investor be willing to pay $100,000 for this annuity?
(c)* What would be the highest applicable discount rate for an investor to be willing to pay $100,000 for this annuity?

4.24. A firm has purchased a piece of equipment for $10,000, which will be financed by a five-year loan accumulating interest at an annual rate of 10%. The loan will be amortized over the five-year period with equal annual payments. What will be the amount of the annual payment?

4.25.* Demonstrate how to derive an expression to determine the present value of a growing annuity.

4.26.* What would be the present value of a 50-year annuity whose first cash flow of $5,000 is paid in ten years and whose final (50th) cash flow is paid in 59 years? Assume that the appropriate discount rate is 12% for all cash flows.

4.27. Suppose that an investor has the opportunity to invest in a stock currently selling for $100 per share. The stock is expected to pay a $1.80 dividend next year (at the end of year one). In each subsequent year forever, the annual dividend is expected to grow at a rate of 4%. All cash flows are to be discounted at an annual rate of 6%. Should the stock be purchased at its current price?

4.28. Suppose that an investor has the opportunity to invest in a stock currently selling for $100 per share. The stock is expected to pay a $5 dividend next year (at the end of year one). In each subsequent year until the third year, the annual dividend is expected to grow at a rate of 15%. Starting in the fourth year, the annual dividend will grow at an annual rate of 6% until the sixth year. Starting in the seventh year, dividends will not grow. All cash flows are to be discounted at an annual rate of 8%. Should the stock be purchased at its current price?

APPENDIX 4.A TIME VALUE SPREADSHEET APPLICATIONS

Spreadsheets are very useful for time value calculations, particularly when there are either a large number of time periods or a large number of potential outcomes. Not are most time value formulas easy to enter into cells, but the toolbar the top of the Excel screen should have the Paste Function button ($\boldsymbol{f_x}$) which will direct the user to a variety of time value functions. By left-clicking the Paste Function ($\boldsymbol{f_x}$), the user will be directed to the Paste Function menu. From the Paste Function menu, one can select the Financial sub-menu. In the Financial sub-menu, scroll down to select the appropriate time value function. Pay close attention to the proper format and arguments for entry. Table A1 below lists a number of time value functions which may be accessed through the Paste Function menu along with the example and notes.

While the formulas entered into Table A.1 make use of specialized Paste Functions for Finance, the spreadsheet user can enter his own simple formulas. For example, suppose that the user enters a cash flow in cell A1, a discount rate in cell A2 and a payment or termination period into cell A3. The present value of this cash flow can be found with =A1/(1+A2)^A3 or, in the case of an annuity, with =A1*((1/A2)-(1/(A2*(1+A2)^A3))). Now, enter a deposit amount into cell A1, an interest rate in cell A2 and a payment date or termination date in cell A3. Future values can be found with =A1*(1+A2)^n and =A1*((1/A2)-(1/(A2*(1+A2)^A3)))*(1+A2)^n. These formulas can easily be adjusted for growth, in which a value for cell A4 may be inserted for the growth rate.

Table A1 Time Value Formula Entry and Paste Functions

	=Function Type from f_x	Format	Entry Example	Result	Formula Entry
1	Future Value of Single Deposit or Investment			−1610.51	=−1000*(1+0.1)^5
2	Future Value of Annuity	=FV(i, n, CF)	=PV(0.1,5,100)	−610.51	=−100*((1+0.1)^5-1)/0.1
3	Future Value of Annuity with FV	=FV(i, n, CF, FV, Type)	=PV(0.1,5,100,1000,0)	−2221.02	=−100*((1+0.1)^5-1)/0.1-1000*(1+0.1)^5
4	Future Value of Annuity Due with FV	=FV(i, n, CF, FV, Type)	=PV(0.1,5,100,1000,1)	−2282.07	=−100*((1+0.1)^6-(1+0.1))/0.1-1000*(1+0.1)^5
5	Present Value of Future Cash Flow			−620.92	=−1000/(1+0.1)^5
6	Net Present Value of Series	=NPV(k, Value 0, Value 1, Value 2, etc.)	=NPV(0.1,−100,110)	0.00	=100/(1+0.1)^0-110/(1+0.1)^1
7	Present Value of Annuity	=PV(k, n, CF)	=PV(0.1,5,100)	−379.08	=−100/0.1*(1-1/(1+0.1)^5)
8	Present Value of Annuity with FV	=PV(k, n, CF, FV, Type)	=PV(0.1,5,100,1000,0)	−1000.00	=−100/0.1*(1-1/(1+0.1)^5)-1000/(1+0.1)^5
9	Present Value of Annuity Due with FV	=PV(k, n, CF, FV, Type)	=PV(0.1,5,100,1000,1)	−1037.91	=−100/0.1*(1-1/(1+0.1)^5)*(1+0.1)-1000/(1+0.1)^5
10	Amortized Payment on Loan	=PMT(I, n, PV)	=PMT(0.1,5,1000)	−263.80	=−1000*0.1/(1-1/(1+0.1)^5)
11	Amortized Payment on Loan (Due)	=PMT(I, n, PV, 0, Type)	=PMT(0.1,5,1000,1000,1)	−239.82	=(−1000*0.1)/((1-1/(1+0.1)^5)*(1+0.1))

Notes

For all the functions above (except 1 and 5), one can either use the *fx* format or the Paste Function Menu retrievable from the toolbar above the spreadsheet.

For all the functions, the formula entry method from the right is usable. See notes below on individual function use. Also, notice how the spreadsheet interprets negative and positive cash flows.

1 Just enter Formula from the right to find the future value of a single cash flow. In this example, X = \$1,000, i = 10% and n = 5.
2 The example is for a \$1,000 annuity for 5 years at 10%.
3 The example is for a \$1,000 annuity for 5 years at 10% plus an additional \$1,000 deposited at time 0.
4 Same as 3 but with all cash flows at the beginning of the periods.
5 Just enter Formula from the right to find the present value of a single cash flow. In this example, CF = \$1,000, k = 10% and n = 5.
6 Net Present value of a series of cash flows starting at time 0. Just enter the amounts of the cash flows which can vary from year-to-year.
7 The example is for a \$1,000 annuity for 5 years at 10%.
8 The example is for a \$1,000 annuity for 5 years at 10% plus \$1,000 received at time 5.
9 Same as 9 but with beginning-of-year cash flows.
10 Amortized (mortgage) payment on a \$1,000 loan at 10% for 5 years.
11 Same as 10 but with beginning-of-year payments.